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Review "Graph Homomorphisms: From Structure to Algorithms" by Karolina Okrasa

The topic of this thesis belongs to the area of Graph Theory. A graph can be viewed as a network consisting of nodes (called vertices) and links between pairs of nodes (called edges). Graphs play a central role in Discrete Mathematics and Theoretical Computer Science, as many network properties and discrete optimization problems can be modeled on graphs. A well-known example, which has numerous applications, and which is considered in the PhD Thesis as well, is the classical Graph Colouring problem: for a given graph G, what is the smallest number of colours such that 1) every vertex of G is allocated exactly one colour, and 2) no two vertices of G that are connected by an edge receive the same colour?

The decision version of the Graph Colouring problem is to decide whether a graph G can be coloured with at most k colours for some integer k. This problem is a special case of the more general Graph Homomorphism problem: the topic of this thesis.

A homomorphism from a graph G to a graph H is a mapping f from the vertex set of G to the vertex set of H such that whenever two vertices u and v are connected by an edge in G, then their images f(u) and f(v) must be connected by an edge in H. If H is the complete graph on k vertices (that is, a graph with an edge between every pair of vertices) then a homomorphism from G to H (if it exists) is, in fact, a k-colouring of G.

A highly natural question is whether and how algorithmic results for Graph Colouring can be generalized to Graph Homomorphism. This question is addressed in the PhD Thesis, which considers in first instance the classical problem Hom(H). In this problem, the graph H is some fixed graph that does not belong to the problem input and the question is: for a given graph G, does there exist a homomorphism from G to H? The famous Hell–Nešetřil Theorem, proven more than three decades ago, is that Hom(H) is polynomial-time solvable if H is a bipartite graph and NP-complete otherwise. If a graph problem is NP-complete, it is natural to restrict the input to some special graph class to increase one's understanding of the computational hardness of the problem. This often requires new insights into the structure of the class first, which one then hopes to exploit algorithmically. So, as the title of the PhD Thesis says, we must go "*from structure to algorithms*".

In recent years, this line of research has been extensively pursued for the Graph Colouring problem. This PhD Thesis adds to the existing body of work by proving several significant results for Hom(H) and also for LHom(H). In the latter problem variant, which also has been well studied in the literature, every vertex u of the input graph G comes with its own list L(u) that is a subset of the vertices of the fixed target graph H. The question is whether there exists a homomorphism from G to H, such that each vertex u of G is mapped to a vertex of L(u).

Chapter 1 is a nice introduction to the Graph Colouring and Graph Homomorphism problems and some well-known graph width parameters, which play an important role in the remainder of the thesis. It reads very well, gives all the relevant results for the Graph Colouring problem under input restrictions, and also contains other useful and appropriate literature references. Moreover, it presents a clear overview on the results in the thesis, several chapters of which have already been published in prestigious venues, such as SIAM Journal on Computing, STACS, ESA and ICALP.

Chapter 2 is a short chapter containing basic graph terminology (the notion of clique-width is illustrated with a nice figure). Chapter 3 is a more technically involved chapter that defines many concepts related to graph homomorphisms. It includes quite a few useful observational statements, which are either proven or taken from the literature. This is a lengthy chapter and illustrates that even the "set up" already requires a considerable level of technical depth. It also states some (sensible) conjectures originating from the literature, which if true have several important consequences, as shown in later chapters.

In Chapter 4, the computational complexity of Hom(H) is studied for graphs of bounded clique-width. Assuming an optimal clique-width expression of the input graph G is given, the main result of this section is that for every target graph H that is a non-trivial projective core, there exists an algorithm for Hom(H) with time complexity s^c * poly(n), where s is the signature number of H and c is the clique-width of G, together with a matching lower bound. This result complements a previous (and an also very strong) result for treewidth of the author of the PhD Thesis.

Chapter 5 is named a "toolbox" for list homomorphisms, and the role of this chapter is like the role of Chapter 3 for homomorphisms. In particular, a number of relevant technical lemmas and their proofs can be found here. Chapter 6 contains similar results (algorithm and lower bound) for LHom(H) for graphs of bounded treewidth, as for Hom(H) restricted to graphs of bounded treewidth and clique-width. As expected, the conditions on the graph H are different.

Chapter 7 is a little different from the previous chapters. In this chapter, recent results on List 3-Colouring for Pt-free graphs are generalized to LHOM(H) for Pt-free graphs (graphs that do not contain a path on t vertices as an induced subgraph), assuming H has a certain property (an n^{O(log^2 n)}-time algorithm is given). It is also proven that if H does not have this property, then there exists an integer t such that LHOM(H) on Pt-free graphs is NP-complete and under ETH does not allow a subexponential-time algorithm. Afterwards, a subexponential-time algorithm is given for LHom(H) restricted to S_{t,t,t}-free graphs, which form a larger graph class than Pt-free graphs. Hence, the condition on H needed some adjustment. Again, this result is accompanied with a corresponding hardness result for the case when H does not satisfy the property.

Finally, Chapter 8 summarizes other important contributions of the author of the PhD-Thesis in this area that were not fully included in the thesis.

This PhD Thesis makes an outstanding contribution to the state-of-art in the field of graph algorithms and their complexity. The thesis is very well structured. It is also well written (my recommendation would be though to do another spell/grammar check and make the list of references consistent). The results are very strong, and their proofs are highly non-trivial, requiring novel insights into both the structure of the graph classes considered and the graph problems themselves.

Given that research on the (List) Graph Colouring problem for special graph classes is still ongoing, the study of the (List) Graph Homomorphism problem is also very timely. It is likely that this will turn out to be an influential PhD Thesis that will make a long-standing impact in the area (especially as some interesting open problems have been identified). It is laudable that all the algorithms in the PhD Thesis are matched with corresponding lower bounds or corresponding hardness results. This illustrates the careful and thorough way the research has been conducted, leading to a complete and well-rounded piece of research.

In summary, this is an outstanding PhD Thesis in Theoretical Computer Science. The author clearly demonstrates to be able to do high-quality research on an international level, and the dissertation clearly meets the standards of a PhD Thesis.

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