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A report on the PhD thesis of
mgr Przemysław Kosewski: "Kołmogorov's Model of
Turbulence – Mathematical Analysis

The PhD thesis by mgr Przemysław Kosewski has been prepared under the supervision of Professor Ewa Zadrzyńska and Dr. Adam Kubica. The main subject of the thesis is the analysis of the Kołmogorov Turbulence Model from the perspective of the existence of regular solutions and a weak approach. The model is a variation of the classical incompressible Navier-Stokes equations and is defined as follows:

$$
\begin{gathered}
v_t + \operatorname{div}(v \otimes v) - \nu \operatorname{div}\left(\frac{b}{\omega} D(v)\right) + \nabla p = 0, \\
\omega_t + \operatorname{div}(\omega v) - \kappa \operatorname{div}\left(\frac{b}{\omega} \nabla \omega\right) + \kappa \omega^2 = 0, \\
b_t + \operatorname{div}(b v) - \kappa \operatorname{div}\left(\frac{b}{\omega} b\right) + b\omega = \kappa \frac{b}{\omega} |D(v)|^2, \\
\operatorname{div} v = 0.
\end{gathered}
$$

Here, v and p represent the velocity and pressure of the fluid, while b and ω are scalars associated with the mean turbulent kinetic energy. This system is well-known in the mathematical community and has been thoroughly analyzed.

The thesis, spanning almost 200 pages, is divided into 5 chapters along with an appendix.
Chapter 1: Introduction serves as an excellent starting point for the thesis, where notation and fundamental concepts are reviewed to facilitate comprehension. This chapter aids in simplifying the reading experience.

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Chapter 2: Local in-time solutions with initial data in H^2 represents a rigorous and standard approach.
Chapter 3: Global in time solutions for small data naturally follows from the content in Chapter 2 under subtle control of the factor b/ω.
Chapter 4: The investigation into the existence and uniqueness of local in-time solutions in H^s builds upon the foundations laid in Chapter 2, demonstrating improvement.
Chapter 5 delves into the existence of weak solutions.
Each of these chapters begins with an introduction, typically comprising only 5 pages. In my opinion, this introduction is rather brief. A more comprehensive introduction would enhance the understanding of the respective chapters.

Chapter 2 is built upon the work of Kosewski and Kubica as presented in their article titled 'Local in Time Solution to Kolmogorov's Two-Equation Model of Turbulence' published in Monatsh. Math. 198 (2022), no. 2, 345–369. In this chapter, the authors commence with H^2 initial data on a torus in 3D and employ the Galernik method to construct reasonably regular solutions. This task, while laborious, is executed with precision and correctness.

Chapter 3 draws upon the work of Kosewski and Kubica, as documented in their article 'Global in Time Solution to Kolmogorov's Two-Equation Model of Turbulence with Small Initial Data,' published in Results Math. 77 (2022), no. 4, Paper No. 163, spanning 31 pages. While building upon the results from Chapter 2, this chapter takes a closer look at the global analysis over time, necessitating a distinct consideration of coefficients associated with the potential degeneration of elliptic operators. Notably, this chapter presents some interesting estimates for these bounds. It exemplifies the Candidate's strong grasp of classical techniques within this field.

Chapter 4 is based on the research presented by Przemysław Kosewski in the paper titled 'Local Well-Posedness of Kolmogorov's Two-Equation Model of Turbulence in Fractional Sobolev Spaces,' which can be found on arXiv:2212.11391. This chapter represents an enhancement of the local existence theorem for fractional spaces H^s. The construction method employed relies on the Galernik approach, with the problem being formulated on a torus once again. To obtain the appropriate sequence of approximations, the original model requires some modification. Once the existence is established, the author demonstrates that the derived function indeed serves as a solution to the original problem, albeit on a limited time interval.

One might naturally wonder why the author did not utilize the results from Chapter 2, which could have expedited the process and simplified the notation. However, this part of the work delves into more analytically advanced territory, necessitating greater attention to fractional regularity. Special estimates for certain commutators are required, as elucidated in the Appendix. Nevertheless, the final outcome still resides within the realm of classical parabolic-type systems theory.

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The attainment of maximal regularity in H^s spaces is achieved through energy methods, albeit through a somewhat laborious process.

This raises the question of whether a more generalized approach based on the L^p framework, as discussed in reference [1] from the thesis, could be considered for future research.

Chapter 5 is based on the paper [28] from the thesis. This paper can be seen as an attempt to refine the techniques presented in [8] by Bulicek and Malek, which were originally applied to a system with boundary constraints. The primary objective is to establish the existence of weak solutions for the Kołmogorov system, as stated in Theorem 5.1.1. Notably, there are significant distinctions between the proofs, and even for the main theorems obtained, some of them result from the utilization of the approach outlined in [36] by Milke and Naumann. These differences arise from varying behaviors of the functions b and ω. Regrettably, the proof is intricate and lengthy, without a summary or sketch, which could have facilitated comprehension.

From an analytical standpoint, this chapter represents the most advanced portion of the work. It's unfortunate that the author did not consider modifying or generalizing the system to yield novel mathematical insights. In this chapter, the author demonstrates a high level of expertise in the techniques of weak solution theory. The proof involves several steps of approximation and a deep understanding of nonlinear analysis methods. Nonetheless, the core scheme follows a similar structure as that in [8].

Summarizing, the thesis presents a coherent set of results pertaining to the issue of existence within the Kołmogorov system. It addresses both local and global temporal existence of regular, unique solutions, as well as includes a section on weak solutions. The Candidate demonstrates a profound understanding of the classical theory of parabolic-type systems, grounded in the energy approach in H^s spaces, where s can be both real and integer. This comprehension of the methods and the system's structure enables the Candidate to establish a global temporal existence result.

The section concerning weak solutions showcases their knowledge of advanced methods in non-linear analysis. My critique pertains to the choice of results. The solutions presented in Chapter 2 are a subset of those derived in Chapter 4; indeed, it suffices to take $s = 2$. Regarding Chapter 5, a natural question arises regarding the selection of the problem. Why has the exact same system as in [8] been considered? Furthermore, there is no discussion of the system's properties and its solutions, which were present in [8]. In a Ph.D. thesis, one would anticipate a more in-depth analysis of the state of the art.

Once again, I wish to emphasize the brevity of the introduction. Providing only estimates does not encompass the entirety of mathematics. Lastly, I am curious as to why the author did not utilize the results of maximal regularity for classical solutions, which could have potentially simplified

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matters and made the spaces more intriguing.

Considering both the strengths and weaknesses of the thesis, I have decided to accept it.

I hereby declare that all the conditions specified in Article 186 and Article 187 of the Higher Education and Science Act (Journal of Laws 2021, item 478) have been met for the award of the doctoral degree in the field of mathematical sciences in the discipline of mathematics. I advocate for the acceptance of the dissertation and the admission of the Candidate to the further stages of the initiated doctoral procedure.

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