REPORT ON THESIS OF MR. TOMASZ PENZA "SUFFICIENT CONDIITONS FOR A MALTSEV PRODUCT OF TWO VARIETIES TO BE A VARIETY"

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The work of Mr. Penza in pursue of PhD degree is devoted to the study of Maltsev product of two varieties.

The Thesis is presented in the document subdivided into 10 sections, including the Introduction and finishing up with the list of references.

The main results of the Thesis are developed in Sections 8 and 9, and the earlier sections introduce and study notions that are needed in formulation and the proof of main results.

The Introduction is important for setting up the context and showing the progress of the investigation prior to the work of the candidate.

The definition of the product of two varieties by A.I. Maltsev, as a generalization of specific constructions of group extensions, was inspired by work of H. Neumann. It appeared in a seminal paper by Maltsev in Siberian Math Journal in 1967. While this operation applied to prevarieties and quasivarieties produces same type of classes, it is not true for varieties, since resulting class might not be closed under homomorphic images. So Maltsev established the first sufficient condition for the product to be a variety: these could be subvarieties of congruence-permutable variety, where algebras have a unique idempotent element. These include varieties of groups, rings and Lie algebras, while does not include variety of lattices, for example.

The Introduction in the Thesis describes work done since inception of the product operation, including quite recent results of others, who investigated sufficient conditions for the product of two varieties to be a variety. One of them involves the variety of lattices, thus, does not relate to congruence permutability.

The direct search on MathSciNet with the key words "Maltsev product of varieties" shows 120 hits with papers since 1978. Some 52 more show, when using different spelling "Mal'cev".

While it is not directly related to the main question of the dissertation, some publications, such as [7], could be mentioned in the Introduction, since Maltsev product play the central role there as well. The authors study 11 families of varieties satisfying some Maltsev conditions, and find that those with congruence permutability, modularity or distributivity may lose the property when applying Maltsev product (and generating a variety afterward, when needed.) This reference is mentioned in section 4 of the Thesis, which is dealing with identities of Maltsev product, thus, automatically considers homomorphically closed classes, even if Maltsev product does not produce a variety.

In our opinion, the candidate could also devote more attention in the Introduction to the detailed presentation of main results of sections 8 and 9, especially since some results appear only in this Thesis, and they generalize already published results mentioned in the references.

Section 2 is devoted to preliminaries, thus, prepares the base of required notation and terminology. Here subsection 2.4 contains important observations related to Maltsev

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product and replica congruence of any algebra in the product, with respect to the variety, which is the second factor in the product.

Starting from section 3, the author considers important new term generalizing the concept of an idempotent element of algebra, which is called *a term idempotent*. This concept also allows to generalize idempotent varieties: term x is term idempotent in idempotent varieties.

In Proposition 3.6 varieties with a term idempotent are described exactly as those for which any algebra has a congruence class as a subalgebra, with respect to any congruence that forces the factor-algebra to be in the variety. This property will appear in the second factor of Maltsev product, in the forthcoming sufficient condition of section 8. Part of this statement appeared earlier in the literature.

Important varieties with term idempotent are so-called *polarized* varieties, where algebras have a unique idempotent and exactly one congruence class of algebra will be a subalgebra. These include groups and rings and were considered by Maltsev in his seminal paper.

Section 4 is devoted to description of the base of identities in the variety generated by Maltsev product of two varieties, and it involves term idempotents of second factor in Maltsev product. The main result here is Theorem 4.2. Important consequence is that the variety generated by Maltsev product will produce the whole class of algebras of given type, if the second factor does not have term idempotents.

Part of this section is also devoted to the investigation of *regularity* and *irregularity* in Maltsev product compared to its factors. This section finishes with the claim about *robustness* of two families of varieties: the term *robust* was introduced in reference publication [7].

Section 5 studies *term idempotent* identities, which are always involve term idempotents on both sides of equality. The main result of this section is Proposition 5.3 that describes the conditions on variety that equivalent to having all consequences of idempotent identities be also idempotent identities. Surprisingly, not always the consequences will be idempotent and they depend on subalgebra of all idempotents being a sink of algebra. For polarized varieties, this is equivalent to have a unique idempotent as a one-element sink, in every algebra of the variety (Proposition 5.4).

In section 6, the author proceeds into the study of varieties whose identities are term idempotent. The definition calls a variety *term idempotent*, if all its non-trivial identities are term idempotent. In particular, idempotent varieties are such, as well as the variety of all algebras of given type. The latter gives considerable free way for establishing results for *absolute* Maltsev product in this largest variety.

This section could be considered as a work of its own value, given a collection of results describing different aspects of *term idempotent* varieties. Of a note, they form a complete sublattice of the lattice of all varieties of a given type. This allows to define a closure operator and a kernel operator on the lattice of all varieties, with term idempotent varieties as outputs. It is not surprising that term idempotent varieties are not closed by taking subvarieties, but it is surprising that homomorphic closure of a Maltsev product does not preserve the property of varieties to be term idempotent (Counterexample 6.20).

Section 7 is devoted to the study of replica congruences with respect to a potential second factor in Maltsev product. When this factor is term idempotent, the congruence classes of any algebra by replica congruence are either a subalgebra or a singleton. This is a generalization from idempotent varieties, when each congruence class would be a subalgebra. Important technical observation is that replica congruence is a transitive closure of a special binary operation on algebra, defined in (7.1) at the start of the section. The key result is then proved in Theorem 7.3.

Overall, it is a bit harder to follow the development in section 7, given a flurry of technical results that all relate to replica congruence of an algebra with respect to the second factor in Maltsev product, under additional conditions. This section would benefit from a small paragraph that complements the description of this section given in the Introduction, providing a guidance to the reader, in anticipation of main results coming in sections 8 and 9.

Direct check shows that section 8 makes reference to Theorem 7.3 and Propositions 7.10 and 7.12. Proposition 7.10 describes condition under which replica congruence is 3-comutable with any congruence on algebra. In the following section 9 there are references to Proposition 7.13 and Theorem 7.14, covering a special case when the meet of two varieties is trivial.

In section 8, the author builds on the technical development of section 7 to formulate new technical conditions $P(n)$ related to joins of replica congruence on algebra, associated with second factor on Maltsev product, with another congruence on algebra. These conditions are in the foundation of the sufficient property for the Maltsev product of two variety to be a variety. It is formulated in Proposition 8.1.

Then technical Propositions 7.10 and 7.12 are used to formulate Maltsev type condition in Theorem 8.6, for the case when the second factor in a Maltsev product is term idempotent. Note that this result generalizes earlier Theorem published in [19].

The last section comes, when the author is poised to reap the rewards given by main results of section 8. For example, earlier result of C. Bergman in [1] claimed that Maltsev product of two idempotent varieties is a variety, provided their join is congruence-permutable variety. In Theorem 9.5, the candidate is able to weaken requirement in sufficient condition: second factor needs to be just term idempotent and the join of varieties is 3-permutable (in particular, it is congruence modular). This is also a new result, which produces 3 Corollaries. Two more results are new: Propositions 9.11 and 9.21, as well as several new Examples. In particular, Example 9.13 dealing with Maltsev product of variety of lattices with some variety of semigroups, whose second binary operation is identical to the first.

Overall, the work of the candidate shows excellent command of intricate concepts and examples that allows to build a complicated network of results extending the previous knowledge about Maltsev product of varieties. This work has theoretical importance within the area of universal algebra, with possible applications in studies of algebras that fit some requirements.

Clearly, the candidate was able to advance some earlier results of his with collaborators, referenced in [3], [18] and [19]. The key results of section 8 employ genuine argument which does not seem to mimic existing strategies. Section 9 could be considered as extended discussion of achievement of previous section showing how broadly the new conditions could be applied. Given the breadth of the results achieved in the topic of considerable importance, I would recommend this dissertation to be distinguished.

The Thesis is well written and practically free of misprints or small errors. I only noticed in the final paragraph of proof of Theorem 7.2, where $a = u(c_1, \ldots, c_n)$ and $b = v(c_1, \ldots, c_n)$, in the next line $u(a_1, \ldots, a_n)$ has to be replaced by $u(c_1, \ldots, c_n)$, and, similarly, to the end of the extended equality.

It is commendable to the candidate to include into Thesis some related open questions. Maybe, the candidate should think about broader directions following this research. Some interesting aspects of the topic which could be worth pursuing:

- Allow signature of algebraic systems to include predicate symbols, and defining sentences $\forall \mathbf{x} P(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$, for n-ary predicate P, besides identities. How will this affect the general picture of Maltsev products? Note that it is Maltsev's legacy to study algebraic systems of mixed type, thus, including both: operations and relations.

- Investigate the lattice of term idempotent varieties, together with operations of the closure and the kernel on the lattice of varieties, which was initiated in Section 6.
- What is known about associativity of Maltsev product, as well as operation of the *right division*, which is inverse to operation of the product? Maltsev wrote about the latter in the 4th section of his paper [15].