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Ph.D. Thesis

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Infinitesimal Generators of Quadratic Harnesses

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Abstract

Quadratic harnesses are Markov polynomial processes with linear conditional expectations and quadratic conditional variances with respect to past-future filtrations. Typically, they are determined by five numerical constants: η, θ, τ, σ, and q, hidden in the form of the conditional variances. In the thesis, we derive infinitesimal generators of these processes, extending the previously known results.

The infinitesimal generator of the quadratic harness is related to a solution of a q-commutation equation in the algebra $\mathcal{Q}$ of infinite sequences of polynomials. The coordinates of the desired solution satisfy a three-term recurrence, defining a system of orthogonal polynomials. The corresponding moment functional uniquely determines the infinitesimal generator and, in certain cases, can be expressed as an integro-differential operator (acting on polynomials or bounded continuous functions with bounded continuous second derivatives) with an explicit kernel, where the integration is with respect to a probabilistic orthogonality measure.

Keywords: polynomial processes, quadratic harnesses, infinitesimal generators, orthogonal polynomials, algebra of polynomial sequences, three-term recurrence

Streszczenie

Kwadratowe harnessy to procesy Markowa będące jednocześnie procesami wielomianowymi o liniowych warunkowych wartościach oczekiwanych i kwadratowych wariancjach warunkowych względem przeszło-przyszłej filtracji. Są one zwykle określone poprzez pięć stałych numerycznych η, θ, τ, σ i q, które występują w postaci warunkowej wariancji. W rozprawie doktorskiej znajdziemy generatory infinitezymalne tych procesów, rozszerzając znane wcześniej wyniki.

Generator infinitezymalny kwadratowego harnessu powiązany jest z rozwiązaniem równania q-komutacyjnego w algebrze $\mathcal{Q}$ nieskończonych ciągów wielomianów. Współrzędne szukanego rozwiązania spełniają formułę trójczłonową, więc definiują rodzinę wielomianów ortogonalnych. Odpowiadający im funkcjonał momentowy jednoznacznie wyznacza generator infinitezymalny, który w pewnych sytuacjach można przedstawić jako operator całkowo-różniczkowy (działający na wielomianach lub ograniczonych funkcjach ciągłych z ciągłą i ograniczoną drugą pochodną) z jawnym jądrem, gdzie całkowanie odbywa się względem probabilistycznej miary ortogonalizującej.

Słowa kluczowe: procesy wielomianowe, kwadratowe harnessy, generatory infinitezymalne, wielomiany ortogonalne, algebra ciągów wielomianów, formuła trójczłonowa

Contents

Introduction

Harnesses, introduced in the 1960s, were proposed to model long-range misorientation in the crystalline structure of metals, see [34]. These objects have been extensively analyzed, even in various abstract settings; however, they are defined using only a first-order conditional structure.

In [18], analogous objects with a specified second-order conditional structure have been introduced. These processes, known as quadratic harnesses, are characterized by five numerical constants. Some well-known examples of quadratic harnesses include Wiener, Poisson, and Gamma processes.

Transition probabilities of quadratic harnesses can be expressed as the orthogonality measures of some polynomial sequences associated with Askey-Wilson polynomials, see [22]. The connection to the orthogonal polynomials is also reflected in free probability ([3], [4]), and free quadratic harnesses can serve as classical counterparts of non-commutative processes.

Furthermore, the relationship between quadratic harnesses and asymmetric simple exclusion processes (ASEPs) has attracted considerable interest. It turns out that the joint generating function of the invariant measure of ASEP can be expressed in a concise form using the joint moments of the corresponding quadratic harness, as shown in [25]. This fact is often applied to analyze the asymptotic behavior of ASEP as the number of sites tends to infinity. The form of the infinitesimal generator of quadratic harness is helpful in analyzing some of these asymptotics, see Section 4.2 of [25].

Extensive research has been conducted on quadratic harnesses over the years, covering various aspects, such as their construction, uniqueness, and infinitesimal generators. However, a universal method for obtaining a formula for the infinitesimal generator in all possible cases has not yet been proposed. This thesis aims to fill this gap by generalizing the previously known cases.

The thesis is divided into two parts. The first part is an introduction to the topic. It also presents a comprehensive proof of the main result along with its conclusions. The second part focuses on the analysis of infinitesimal generators of quadratic harnesses for specially chosen parameters.

Let us now introduce a notation that will be used in the thesis.

Notation

Throughout the thesis, we use the following notation: $\mathbb{N}$ denotes the set of natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers. Additionally, we define $\mathbb{N}_0$ as the set $\mathbb{N} \cup \{0\}$.

Let $A \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N}$. By $\mathcal{C}(A)$, we denote the set of continuous functions $f : A \to \mathbb{R}$, and by $\mathcal{C}^1(A)$, we refer to the set of continuously differentiable functions, i.e., functions $f : A \to \mathbb{R}$ that are continuously differentiable in the interior of A and can be continuously extended along with all their partial derivatives to the boundary of A. Similarly, $\mathcal{C}^2(A)$ represents the set of twice continuously differentiable functions, where $f : A \to \mathbb{R}$ satisfies $f \in \mathcal{C}^1(A)$ and all its (continuously extended to the boundary) partial derivatives are also in $\mathcal{C}^1(A)$.

Furthermore, we consider and fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and use standard mathematical notation related to the probability theory. For example, the expected value with respect to $\mathbb{P}$ is denoted as $\mathbb{E}$. Moreover, we will consider a stochastic process $(X_t)_{t \geqslant 0}$ defined on this probability space. The support of the one-dimensional distribution of the process at time $t \geqslant 0$ will be denoted as $\operatorname{supp}(X_t)$.

Some additional notations, which do not necessarily coincide with standard conventions, will be introduced and explained as needed throughout the thesis.

Chapter 1

Preliminaries

In the thesis, we study quadratic harnesses, which are Markov processes with specified first and second conditional moments. Our main objective is to determine the form of infinitesimal generators of these processes, particularly when they act on polynomials. To achieve this, we will introduce all relevant concepts here.

1.1. Harnesses

In [34], several concepts of processes that enjoy certain properties of conditional expectations have been introduced. Of particular interest among these concepts are harnesses. Let us consider an integrable stochastic process $(X_t)_{t\geqslant 0}$ with past-future filtrations defined as

$$\mathcal{F}_{s,u} := \sigma\{X_t : t \in [0,s] \cup [u,\infty)\}, \qquad 0 \leqslant s < u.$$

Definition 1.1.1. We say that $(X_t)_{t\geqslant 0}$ is a *harness* if it satisfies the following linearity of regression property for all $0 \leqslant s < t < u$:

$$\mathbb{E}(X_t|\mathcal{F}_{s,u}) = \tfrac{u-t}{u-s}X_s + \tfrac{t-s}{u-s}X_u. \tag{1.1}$$

Harnesses have been encountered by many authors in their research, often without realizing it. For instance, [38, Theorem 2] states that every integrable Lévy process is a harness. Notably, the Wiener process and the Poisson process are examples of harnesses.

Harnesses have many interesting properties and applications, see e.g. [8], [33], [35].

We are particularly interested in the case when the second moments of $(X_t)_{t\geqslant 0}$ are finite. To avoid ambiguity (see the comment below Proposition 2.1 in [18]), we assume throughout the thesis that

$$\mathbb{E}(X_t) = 0 \qquad \text{and} \qquad \mathbb{C}\mathrm{ov}(X_s, X_t) = \min\{s,t\}, \qquad s,t \geqslant 0. \tag{1.2}$$

Under this assumption, harnesses satisfy martingale and reverse martingale conditions. That is, for all $0 \leqslant s < t$,

$$\mathbb{E}(X_t|\mathcal{F}_{\leqslant s}) = X_s \qquad \text{and} \qquad \mathbb{E}(X_s|\mathcal{F}_{\geqslant t}) = \tfrac{s}{t}X_t,$$

where $\mathcal{F}_{\leqslant s} := \sigma\{X_r : 0 \leqslant r \leqslant s\}$ and $\mathcal{F}_{\geqslant s} := \sigma\{X_u : u \geqslant s\}$. Consequently, the following limits exist almost surely:

$$\lim_{t\to\infty} \tfrac{1}{t}X_t = 0 \qquad \text{and} \qquad \lim_{t\to 0^+} X_t = 0. \tag{1.3}$$

Moreover, it is easy to verify that if $(X_t)_{t\geqslant 0}$ is a harness, then a time inverse process $(tX_{1/t})_{t\geqslant 0}$ is also a harness. The time inverse process at time 0 should be understood according to the almost sure limit, i.e., it is equal to zero almost surely. Proofs of the aforementioned facts can be found in [18].

1.2. Quadratic harnesses

Let us consider a square integrable stochastic process $(X_t)_{t\geqslant 0}$. The linearity of conditional first moments for harnesses suggests the form of conditional second moments.

Definition 1.2.1. We say that $(X_t)_{t\geqslant 0}$ is a *quadratic harness* if it is a harness and for all $0 \leqslant s < t < u$

$$\mathbb{E}(X_t^2|\mathcal{F}_{s,u}) = A_{t,s,u}X_s^2 + B_{t,s,u}X_sX_u + C_{t,s,u}X_u^2 + D_{t,s,u}X_s + E_{t,s,u}X_u + F_{t,s,u}, \tag{1.4}$$

where $A_{t,s,u}, \ldots, F_{t,s,u}$ are some deterministic functions depending only on times s, t and u.

Under assumption (1.2), we get

$$A_{t,s,u}s^2 + B_{t,s,u}s + C_{t,s,u}u + F_{t,s,u} = t.$$

Moreover, it turns out that these coefficients can be written explicitly in terms of certain five numerical constants, as stated in [18, Theorem 2.2] (generalized later in [26, Theorem 4.4]):

Theorem 1.2.1. *Let* $(X_t)_{t\geqslant 0}$ *be a quadratic harness satisfying* (1.2). *Suppose that* $F_{t,s,u}\neq 0$ *for all* $0\leqslant s<t<u$ *and that* 1, X_s, X_t, X_s^2, X_t^2, *and* X_sX_t *are linearly independent as functions on* Ω *for all* $0<s<t$. *Then there exist*

$$\eta,\theta\in\mathbb{R},\quad \sigma,\tau\geqslant 0\quad \textit{and}\quad q\leqslant 1+2\sqrt{\sigma\tau} \tag{1.5}$$

such that for all $0<s<t<u$

$$\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{s,u})=F_{t,s,u}K\left(\tfrac{X_u-X_s}{u-s},\tfrac{uX_s-sX_u}{u-s}\right),$$

where $F_{t,s,u}=\frac{(u-t)(t-s)}{u(1+\sigma t)+\tau-qs}$ *and* $K(x,y):=1+\theta x+\tau x^2+\eta y+\sigma y^2-(1-q)xy$.

Moreover, taking the limits $u\to\infty$ and $s\to 0$, respectively, leads to the following identities:

$$\begin{aligned}\mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{\leqslant s})&=\tfrac{t-s}{1+\sigma s}(\sigma X_s^2+\eta X_s+1), \qquad (1.6)\\ \mathbb{V}\mathrm{ar}(X_t|\mathcal{F}_{\geqslant u})&=\tfrac{t(u-t)}{u+\tau}\left(\tau\tfrac{X_u^2}{u^2}+\theta\tfrac{X_u}{u}+1\right),\end{aligned}$$

see [18, (2.27)–(2.28)]. From the form of the conditional variance, we can easily calculate the second conditional moment and obtain formulas for the coefficients appearing in (1.4) as follows:

$$\begin{aligned}&A_{t,s,u}=\tfrac{(u-t)(u(1+\sigma t)+\tau-qt)}{(u-s)(u(1+\sigma s)+\tau-qs)},\ B_{t,s,u}=\tfrac{(1+q)(u-t)(t-s)}{(u-s)(u(1+\sigma s)+\tau-qs)},\ C_{t,s,u}=\tfrac{(t-s)(t(1+\sigma s)+\tau-qs)}{(u-s)(u(1+\sigma s)+\tau-qs)},\\ &D_{t,s,u}=\tfrac{(u-t)(t-s)(\eta u-\theta)}{(u-s)(u(1+\sigma s)+\tau-qs)},\ E_{t,s,u}=\tfrac{(u-t)(t-s)(\theta-\eta s)}{(u-s)(u(1+\sigma s)+\tau-qs)}.\end{aligned} \tag{1.7}$$

In the thesis, we will consider only quadratic harnesses for which all moments exist. In particular, this assumption is satisfied when $\sigma\tau=0$, see [18, Theorem 2.5]. In all known cases when all moments exist, the quadratic harness $(X_t)_{t\geqslant 0}$ is a uniquely determined Markov process with parameters η, θ, σ, τ, and q, compare with the comment below Theorem 2.4 in [18]. Hence, in order to refer to a quadratic harness with the appropriate parameters, we will use $QH(\eta,\theta;\sigma,\tau;q)$.

While every integrable Lévy process is a harness, not every square integrable Lévy process is a quadratic harness. The second-order conditional structure depends on the distribution of the process.

Example 1.2.2. *The Wiener process* $(W_t)_{t\geqslant 0}$ *is a harness satisfying* (1.2) *and*

$$\mathbb{V}\mathrm{ar}(W_t|\mathcal{F}_{s,u}) = \tfrac{(u-t)(t-s)}{u-s}, \qquad 0 \leqslant s < t < u.$$

Indeed, it is obvious that (1.2) *is satisfied. Moreover, by the Markov property we have:*

$$\mathbb{E}(W_t|\mathcal{F}_{s,u}) = \mathbb{E}(W_t|W_s, W_u) \qquad \text{and} \qquad \mathbb{V}\mathrm{ar}(W_t|\mathcal{F}_{s,u}) = \mathbb{V}\mathrm{ar}(W_t|W_s, W_u). \tag{1.8}$$

To find formulas for these expressions, let us consider a characteristic function of the random vector (W, W_s, W_u)*:*

$$\varphi(x, y, z) := \mathbb{E}\exp(ixW + iyW_s + izW_u),$$

where $W := (u-s)W_t - (u-t)W_s - (t-s)W_u$*. Since* $(W_t)_{t\geqslant 0}$ *has independent increments, we get:*

$$\varphi(x, y, z) = \mathbb{E}\exp\left(i(z - x(t-s))(W_u - W_t)\right) \cdot \mathbb{E}\exp\left(i(x(u-t) + z)(W_t - W_s)\right) \cdot \mathbb{E}\exp\left(i(y+z)W_s\right).$$

Applying the formulas for characteristic functions of normal distributions with zero means and variances equal to $u-t$*,* $t-s$*, and* s*, respectively, we obtain*

$$\varphi(x, y, z) = \exp(-(u-t)(t-s)(u-s)x^2/2 - uz^2/2 - sy^2/2 - syz) = \varphi_1(x)\varphi_2(y, z),$$

where $\varphi_1(x) := \exp(-(u-t)(t-s)(u-s)x^2/2)$ *and* $\varphi_2(y, z) := \exp(-uz^2/2 - sy^2/2 - syz)$*. As a result,* W *is independent of* (W_s, W_u) *and* φ_1 *is the characteristic function of* W*. Moreover,*

$$\mathbb{E}W = \varphi_1'(0) = 0 \qquad \text{and} \qquad \mathbb{V}\mathrm{ar}(W) = \mathbb{E}W^2 = \varphi_1''(0) = (u-t)(t-s)(u-s).$$

Therefore, in view of (1.8),

$$\mathbb{E}(W_t|\mathcal{F}_{s,u}) = \tfrac{1}{u-s}\mathbb{E}(W|W_s, W_u) + \tfrac{u-t}{u-s}W_s + \tfrac{t-s}{u-s}W_u = \tfrac{u-t}{u-s}W_s + \tfrac{t-s}{u-s}W_u$$

and

$$\mathbb{V}\mathrm{ar}(W_t|\mathcal{F}_{s,u}) = \tfrac{1}{(u-s)^2}\mathbb{V}\mathrm{ar}(W|W_s, W_u) = \tfrac{1}{(u-s)^2}\mathbb{V}\mathrm{ar}(W) = \tfrac{(u-t)(t-s)}{u-s}.$$

As a result, $(W_t)_{t\geqslant 0}$ *is* $QH(0,0;0,0;1)$*, with the parameters identified from the formula for the conditional variance.*

Moreover, we can show the following:

Example 1.2.3. *Let* $(N_t)_{t\geqslant 0}$ *be a Poisson process with rate* $\lambda > 0$ *and consider*

$$Y_t := \tfrac{N_t - \lambda t}{\sqrt{\lambda}}, \qquad t \geqslant 0.$$

Then $(Y_t)_{t\geqslant 0}$ *is a harness satisfying* (1.2) *and*

$$\mathbb{V}\mathrm{ar}(Y_t|\mathcal{F}_{s,u}) = \tfrac{(u-t)(t-s)}{u-s}\left(1 + \tfrac{1}{\sqrt{\lambda}}\tfrac{Y_u - Y_s}{u-s}\right), \qquad 0 \leqslant s < t < u.$$

Hence $(Y_t)_{t\geqslant 0}$ *is* $QH(0, 1/\sqrt{\lambda}; 0, 0; 1)$*.*

Moreover, in the class of quadratic harnesses we can find also the following processes:
— $QH(0,\theta;0,\tau;1)$ – Lévy-Meixner process [43],
— $QH(0,0;0,0;q)$ – classical version of the q-Brownian motion [14] (free Brownian motion [10] when $q = 0$),
— $QH(\eta,\theta;0,0;q)$ – bi-Poisson process [19] (quantum Bessel process [9] when $q = 1$),
— $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$ – free quadratic harness [20].

With harnesses or quadratic harnesses, certain algebraic structures known as near algebras are associated, which capture changes in conditional expectations that arise from applying the tower property, as shown in [26]. The algebraic methods described in that paper facilitate the analysis of these processes, enabling their parametric description. Similarly, an algebraic language of polynomial sequences has proven to be useful for quadratic harnesses in other contexts as well, see [24].

1.3. Construction of a quadratic harness

The question of existence of quadratic harnesses for a given set of parameters is non-trivial. Some constructions have been carried out for a rather wide range of parameters, see [19], [16], [22], [20], [17], [44], but not for the full range. Moreover, the parameters in the constructions are given in a complicated and abstruse way, especially when $\sigma\tau > 0$, compare with [22, Theorem 1.1].

Quadratic harnesses are typically Markov processes with all moments finite. As it can be found in Section 3.2 of [22], transition probabilities orthogonalize a system of polynomials, specifically Askey-Wilson polynomials with appropriately chosen parameters and linearly transformed arguments.
It is important to emphasize that the support of a quadratic harness must satisfy an additional condition.

Remark 1.3.1. *For all $x \in supp(X_t)$, the inequality*

$$1 + \eta x + \sigma x^2 \geqslant 0$$

holds, as the conditional variance is non-negative almost surely, see (1.6)*.*

In the thesis, we assume that all moments of the quadratic harness we are considering are finite. However, this assumption may not hold in certain cases. For example, there exists a quadratic harness with $\sigma\tau > 0$ and $q > 1$, where $\mathbb{E}|X_t|^{2+\delta} = \infty$ for all $t > 0$ and $\delta > 0$, see [37].
An example of a wide range of parameters for which the corresponding quadratic harness exists is

$$-1 \leqslant q \leqslant 1 - 2\sqrt{\sigma\tau} \qquad \text{and} \qquad 0 \leqslant \sigma\tau < 1, \tag{1.9}$$

see [22, Remark 1.3] and [16, Proposition 4.2]. When (1.9) holds with $\eta = \theta = 0$, the quadratic harnesses have all moments finite. However, it is worth mentioning that the range of η and θ for which the quadratic harness with all moments finite exists may be broader. In the thesis, we will make no additional assumptions on η and θ, other than they are reals.

If all moments are finite, then there exist *martingale polynomials*, see [24, Section 1.2], i.e., there exist polynomials $\{p_n(x;t)\}_{n=0}^{\infty}$ satisfying

$$\mathbb{E}(p_n(X_t;t)|\mathcal{F}_{\leqslant s}) = p_n(X_s;s) \tag{1.10}$$

for every $t > s \geqslant 0$.

Under assumptions (1.9), we have explicit expressions for these polynomials, see [18]. As given in [18, (4.13)], the first three polynomials are:

$$p_0(x;t) = 1, \qquad p_1(x;t) = x, \qquad p_2(x;t) = \tfrac{1}{1+\sigma t}x^2 - \tfrac{(\eta+\theta\sigma)t+\eta\tau+\theta}{(1-\sigma\tau)(1+\sigma t)}x - \tfrac{t}{1+\sigma t}. \tag{1.11}$$

The subsequent polynomials satisfy a three-step recurrence for $n \in \mathbb{N}$:

$$xp_n(x;t) = (\sigma\alpha_{n+1}t+\beta_{n+1})p_{n+1}(x;t)+(\gamma_n t+\delta_n)p_n(x;t)+(\beta_n t+\tau\alpha_n)\omega_n p_{n-1}(x;t), \tag{1.12}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\delta_n\}_{n=1}^{\infty}$ and $\{\omega_n\}_{n=1}^{\infty}$ are some sequences of coefficients given in Theorem 4.1. in [18]. Particularly,

$$\omega_2 = (1+q)\tfrac{(1-\sigma\tau)^2+(\eta+\theta\sigma)(\theta+\eta\tau)}{(1-\sigma\tau)^2[1-\sigma\tau(2+q)]}. \tag{1.13}$$

Lemma 4.2 in [18] ensures that

$$\beta_n > \sqrt{\sigma\tau}\alpha_n \geqslant 0 \qquad \text{and} \qquad \alpha_n \geqslant 0, \qquad n \in \mathbb{N}.$$

Therefore, the coefficient of p_{n+1} in (1.12) is positive for all $t \geqslant 0$, and thus the polynomials $\{p_n(x;t)\}_{n=0}^{\infty}$ are well-defined. Moreover, it is worth mentioning that $\{p_n(\cdot;t)\}_{n=0}^{\infty}$ are orthogonal with respect to the distribution of X_t, $t \geqslant 0$.

For more on martingale polynomials and their relations to the quadratic harness, see [40], [41], [42].

1.4. Quadratic harness as a polynomial process

Since the coefficients in the recurrence (1.12) are affine functions of $t \geqslant 0$, and the coefficient at p_{n+1} is positive, the martingale polynomials $\{p_n(x;t)\}_{n=0}^{\infty}$ satisfy for all $n \in \mathbb{N}_0$:

$$p_n(x;t) = \sum_{k=0}^{n} a_{k,n}(t)x^k,$$

and

$$x^n = \sum_{k=0}^{n} b_{k,n}(t)p_k(x;t),$$

where $a_{k,n}$ and $b_{k,n}$ are some rational functions of t, well-defined for all $t \geqslant 0$, $k = 0, 1, \ldots, n$, hence they are in $\mathcal{C}^1([0,\infty))$. Consequently,

$$\mathbb{E}(X_t^n | X_s) = \mathbb{E}[\mathbb{E}(X_t^n | \mathcal{F}_{\leqslant s}) | X_s] = \sum_{k=0}^{n} b_{k,n}(t)p_k(X_s;s),$$

and

$$\mathbb{E}(X_t^n | X_s = x) = \sum_{k=0}^{n} b_{k,n}(t)p_k(x;s) = \sum_{k=0}^{n} b_{k,n}(t)\left(\sum_{l=0}^{k} a_{l,k}(s)x^l\right) = \sum_{l=0}^{n}\left(\sum_{k=l}^{n} a_{l,k}(s)b_{k,n}(t)\right)x^l.$$

Hence for any polynomial f of degree at most n,

$$\mathbb{E}(f(X_t)|X_s = x)$$

is a polynomial in variable x of degree at most n. Furthermore, the function

$$(s,t) \mapsto \mathbb{E}(f(X_t)|X_s = x)$$

is in $\mathcal{C}^1(\Gamma)$, where

$$\Gamma := \{(s,t) \in \mathbb{R}^2 : 0 \leqslant s \leqslant t\}. \tag{1.14}$$

Processes with these properties are called polynomial processes. Let us recall the formal definition of polynomial processes, based on definitions provided in [1] and [2].

Definition 1.4.1. We say that $(X_t)_{t \geqslant 0}$ is an *m-polynomial process*, $m \in \mathbb{N}_0$, if for any polynomial f of degree at most $k \leqslant m$, the following two conditions hold:

1. $\mathbb{E}(f(X_t)|X_s = x)$ is a polynomial in variable x of degree at most k,
2. $(s,t) \mapsto \mathbb{E}(f(X_t)|X_s = x)$ is in $\mathcal{C}^1(\Gamma)$.

Definition 1.4.2. We say that $(X_t)_{t\geqslant 0}$ is a *polynomial process* if $(X_t)_{t\geqslant 0}$ is an m-polynomial process for all $m \in \mathbb{N}_0$.

Polynomial processes were first introduced by Cuchiero [29] in the time-homogeneous case. In that case, the second condition in the definition of an m-polynomial process can be relaxed to functions belonging to $\mathcal{C}(\Gamma)$ instead of $\mathcal{C}^1(\Gamma)$.
Applications of polynomial processes in finance and insurance mathematics, as discussed in [30], have generated significant interest and extensive research over the past decade. Furthermore, some extensions of polynomial processes to more abstract settings have been proposed in [31], [32], [7], and [6]. In these papers, substantial effort has been dedicated to investigating properties of infinitesimal generators of polynomial processes. Specifically, they aim to use infinitesimal generators in the context of a martingale problem and thus simplify the computation of certain expectations.
Consequently, providing explicit formulas for infinitesimal generators for a wide class of polynomial processes is of considerable interest.

1.5. Infinitesimal generators of quadratic harnesses

Denote by $\{\mathbb{P}_{s,t}(x, \mathrm{d}y) : x \in \mathbb{R}, 0 \leqslant s < t\}$ the transition probabilities of a quadratic harness $(X_t)_{t\geqslant 0}$. Since, in general, $(X_t)_{t\geqslant 0}$ is a non-homogeneous Markov process, we have to consider right and left infinitesimal generators indexed by a time variable $t \geqslant 0$.
We say that $\mathbf{A}_t^+$ is a *weak right infinitesimal generator* if

$$(\mathbf{A}_t^+ f)(x) := \lim_{h\to 0^+} \frac{\mathbb{E}(f(X_{t+h})|X_t=x)-f(x)}{h} = \lim_{h\to 0^+} \int_{\mathbb{R}} \frac{f(y)-f(x)}{h} \mathbb{P}_{t,t+h}(x, \mathrm{d}y) \tag{1.15}$$

for all $t \geqslant 0$ and for all functions f for which this limit exists pointwise. Then we say that f is *in the domain of* $\mathbf{A}_t^+$ and denote this fact as $f \in \mathcal{D}(\mathbf{A}_t^+)$.
Analogously, $\mathbf{A}_t^-$ is a *weak left infinitesimal generator* if

$$(\mathbf{A}_t^- f)(x) := \lim_{h\to 0^+} \frac{\mathbb{E}(f(X_t)|X_{t-h}=x)-f(x)}{h} = \lim_{h\to 0^+} \int_{\mathbb{R}} \frac{f(y)-f(x)}{h} \mathbb{P}_{t-h,t}(x, \mathrm{d}y) \tag{1.16}$$

for $t > 0$ and f as above. The domain of $\mathbf{A}_t^-$ is denoted by $\mathcal{D}(\mathbf{A}_t^-)$. For homogeneous processes, the two operators coincide and do not depend on time t. However, in general, these two operators may not coincide, see [13, Example 2.1].

For quadratic harnesses, the infinitesimal generators act nicely on martingale polynomials. Indeed, the Markov property implies that $\mathbb{E}(p_n(X_t,t)|\mathcal{F}_{\leqslant t-h}) = \mathbb{E}(p_n(X_t,t)|X_{t-h})$, and according to (1.10), we have

$$\begin{aligned}\mathbf{A}_t^-(p_n(x;t)) &= \lim_{h\to 0^+}\frac{\mathbb{E}(p_n(X_t;t)|X_{t-h}=x)-p_n(x;t)}{h}\\ &= \lim_{h\to 0^+}\frac{p_n(x;t-h)-p_n(x;t)}{h} = -\tfrac{\partial}{\partial t}p_n(x;t).\end{aligned}$$

With some additional effort (see Section 1.4 in [24] or Lemma 2.1 in [23]), it can be shown that

$$\mathbf{A}_t^+(p_n(x,t)) = -\tfrac{\partial}{\partial t}p_n(x;t).$$

Hence, by linearity, for any polynomial f, we have $f \in \mathcal{D}(\mathbf{A}_t^-) \cap \mathcal{D}(\mathbf{A}_t^+)$ and

$$\mathbf{A}_t^+ f = \mathbf{A}_t^- f.$$

Thus, when considering polynomials, we can use the same symbol $\mathbf{A}_t$ for both operators. Furthermore, Agoitia-Hurtado proved that for any polynomial process (a quadratic harness in particular), there exists a Banach space of polynomials up to degree $m \in \mathbb{N}_0$ such that pointwise convergence of $\mathbf{A}_t$ implies convergence in the norm of this space. For more details, refer to Proposition 2.2.10 in [1]. Moreover, Lemma 2.2.8 therein provides a formula for $\mathbf{A}_t$ acting on any polynomial f: if

$$\mathbb{E}(f(X_t)|X_s=x) = \sum_{l=0}^{k}\alpha_l^f(s,t)x^l, \tag{1.17}$$

then $\alpha_l^f(s,t) \in \mathcal{C}^1(\Gamma)$ (recall (1.14)) for all $l = 0,1,\dots,k$, and

$$(\mathbf{A}_t f)(x) = \sum_{l=0}^{k}\tfrac{\partial}{\partial t}\alpha_l^f(s,t)x^l|_{s=t}. \tag{1.18}$$

This representation strongly depends on the form of the polynomial f, in particular on the coefficients presented on the right-hand side of (1.17). Our objective is to find

an alternative form of the infinitesimal generator that is independent of f and instead reflects an underlying structure of the stochastic process under investigation.
Over the years, various approaches have been proposed to derive explicit formulas for the infinitesimal generators of quadratic harnesses, with different restrictions on the parameters η, θ, σ, τ, and q, see [11], [5], [15], [23] (generalized later in [25]), and [24]. All these representations lead to an integro-differential operator of the form

$$(\mathbf{A}_t f)(x) = \tfrac{1+\eta x+\sigma x^2}{1+\sigma t} \int_{\mathbb{R}} \tfrac{\partial}{\partial x} \left(\tfrac{f(y)-f(x)}{y-x} \right) \nu_{x,t}(\mathrm{d}y), \tag{1.19}$$

where $\nu_{x,t}$ is a probability measure. Within this approach, in order to determine the infinitesimal generator $\mathbf{A}_t$, one needs to determine the measure $\nu_{x,t}$.

Let us analyze the Wiener process as an example.

Example 1.5.1. *Recall that the Wiener process* $(W_t)_{t\geqslant 0}$ *is a quadratic harness* $QH(0,0;0,0;1)$*, see Example 1.2.2. Since* $(W_t)_{t\geqslant 0}$ *is also a Markov process with independent and stationary increments, we have*

$$\mathbb{E}(W_t^k | W_s = x) = \mathbb{E}((W_t - W_s + x)^k | W_s = x) = \mathbb{E}(W_{t-s} + x)^k$$

for all $t > s \geqslant 0$*. Using the binomial formula, we obtain*

$$\mathbb{E}(W_t^k | W_s = x) = \sum_{l=0}^{k} \binom{k}{l} x^{k-l} \mathbb{E}W_{t-s}^l = \sum_{l=0}^{k} \binom{k}{l} x^{k-l} (t-s)^{l/2} \mathbb{E}W_1^l,$$

where in the last equality we used the fact that $W_{t-s} \stackrel{d}{=} \sqrt{t-s}W_1$*. Since* $\mathbb{E}W_1 = 0$ *and* $\mathbb{E}W_1^2 = 1$*, formula* (1.18) *implies*

$$\mathbf{A}_t(x^k) = \begin{cases} 0, & k = 0, 1, \\ \frac{k(k-1)}{2} x^{k-2}, & k \geqslant 2. \end{cases}$$

This formula can also be obtained by direct calculation, recall (1.15)*. It is worth noting that* $\mathbf{A}_t(x^k)$ *can be rewritten as*

$$\mathbf{A}_t(x^k) = \int_{\mathbb{R}} \tfrac{\partial}{\partial x} \left(\tfrac{y^k - x^k}{y-x} \right) \delta_x(\mathrm{d}y), \qquad k \in \mathbb{N}_0,$$

where δ_x is a Dirac measure concentrated at x. In fact, this holds true for $k = 0, 1$. Moreover, for $k \geqslant 2$, we have

$$\int_{\mathbb{R}} \tfrac{\partial}{\partial x}\left(\tfrac{y^k - x^k}{y - x}\right)\delta_x(\mathrm{d}y) = \sum_{l=1}^{k-1}\int_{\mathbb{R}} l x^{l-1} y^{k-1-l}\delta_x(\mathrm{d}y) = x^{k-2}\sum_{l=1}^{k-1} l = \tfrac{k(k-1)}{2}x^{k-2}.$$

As a result, by linearity, we get

$$(\mathbf{A}_t f)(x) = \int_{\mathbb{R}} \tfrac{\partial}{\partial x}\left(\tfrac{f(y)-f(x)}{y-x}\right)\delta_x(\mathrm{d}y).$$

Thus, for the Wiener process, equation (1.19) *holds with $\nu_{x,t} = \delta_x$.*

Our aim in this thesis is to derive a formula for the measure $\nu_{x,t}$ appearing in (1.19) in the general case of any quadratic harness $QH(\eta, \theta; \sigma, \tau; q)$ satisfying (1.9). This measure will be expressed as the orthogonality measure of a certain family of polynomials that satisfy a three-step recurrence.

1.6. Main result of the thesis

In [44], we extended the algebraic approach from [24] to incorporate the framework of orthogonal polynomials established in [23]. This extension enabled us to close the problem of finding the infinitesimal generators in the case $\sigma = 0$. However, in order to obtain formulas for the infinitesimal generators in all relevant cases, including those that are currently unknown or difficult to derive directly, as will be seen in the thesis, a considerable effort is required to develop the methodology of [24] and [44].
Before we state our main theorem, let us introduce the q-notation:

$$[n]_q = 1 + q + \ldots + q^{n-1} \text{ for } n \in \mathbb{N} \text{ and, by convention, } [n]_q = 0 \text{ for } n \leqslant 0. \tag{1.20}$$

For the sake of brevity, we will also use the following notation:

$$\xi := 1 + q + \sqrt{(1-q)^2 - 4\sigma\tau}, \tag{1.21}$$

and

$$\xi_0 := \tfrac{4(\tau + (1-q)t + \sigma t^2)}{\xi^2}, \qquad \xi_1 := \tfrac{2(1+\sigma t)}{\xi} - 1, \qquad \xi_2 := \tfrac{2(\theta - \eta t)}{\xi}. \tag{1.22}$$

Under assumptions (1.9), we clearly have $\xi > 0$. Consequently, ξ_0, ξ_1, and ξ_2 are well-defined.

Now we are ready to present the main result of the thesis (the notions related to orthogonal polynomial theory, that appear in the statement of the theorem, are explained in the appendix):

Theorem 1.6.1. *Assume* (1.9). *Then the infinitesimal generator of $QH(\eta,\theta;\sigma,\tau;q)$ acting on an arbitrary polynomial f is given by*

$$(\mathbf{A}_t\, f)(x) = \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}\left[\tfrac{\partial}{\partial x}\tfrac{f(y)-f(x)}{y-x}\right],$$

where $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$ is a normalized moment functional for a weak orthogonal polynomial sequence $\{\widetilde{W}_n(\cdot;x,t)\}_{n=0}^{\infty}$ satisfying a three-step recurrence:

$$\begin{aligned} &\widetilde{W}_{-1}(y;x,t) = 0, \qquad \widetilde{W}_0(y;x,t) = 1, \\ &y\widetilde{W}_n(y;x,t) = \widetilde{W}_{n+1}(y;x,t) + \widetilde{a}_n(x)\widetilde{W}_n(y;x,t) + \widetilde{b}_n(x)\widetilde{W}_{n-1}(y;x,t), \quad n \in \mathbb{N}_0, \end{aligned} \tag{1.23}$$

with

$$\begin{aligned} \widetilde{a}_n(x) &:= \xi_0\tfrac{[n]_{\widetilde{q}}[n+1]_{\widetilde{q}}}{1+\xi_1[2n+1]_{\widetilde{q}}}\left(\sigma(\widetilde{\gamma}_{n+1}(x)+\widetilde{\gamma}_n(x))+\eta\right)+\widetilde{\gamma}_{n+1}(x), \\ \widetilde{b}_n(x) &:= \xi_0\tfrac{[n]_{\widetilde{q}}[n+1]_{\widetilde{q}}(1+\xi_1[n-1]_{\widetilde{q}})(1+\xi_1[n]_{\widetilde{q}})}{(1+\xi_1[2n-1]_{\widetilde{q}})(1+\xi_1[2n+1]_{\widetilde{q}})}\left(1+\widetilde{\gamma}_n(x)(\sigma\widetilde{\gamma}_n(x)+\eta)\right), \end{aligned}$$

where

$$\widetilde{\gamma}_n(x) := \frac{x\widetilde{q}^n+\eta\xi_0[n]_{\widetilde{q}}^2+\xi_2[n]_{\widetilde{q}}(1+\xi_1[n]_{\widetilde{q}})}{1+\xi_1[2n]_{\widetilde{q}}}$$

and

$$\widetilde{q} := \tfrac{4(q+\sigma\tau)}{\xi^2}. \tag{1.24}$$

Moreover, if $1+\eta x+\sigma x^2 > 0$, then we have an integro-differential representation:

$$(\mathbf{A}_t\, f)(x) = \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\int_{\mathbb{R}}\tfrac{\partial}{\partial x}\left(\tfrac{f(y)-f(x)}{y-x}\right)\,\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y),$$

where $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$ is the probabilistic orthogonality measure for the polynomials $\{\widetilde{W}_n(\cdot;x,t)\}_{n=0}^{\infty}$.

Note that the polynomials $\{\widetilde{W}_n(y;x,t)\}_{n=0}^{\infty}$ depend on t through the parameters ξ_0, ξ_1, and ξ_2.

Furthermore, every x from the support of X_t, $t \geqslant 0$, satisfies $1 + \eta x + \sigma x^2 \geqslant 0$, as stated in Remark 1.3.1. If the inequality is strict, Theorem 1.6.1 gives an integro-differential representation. In the case of equality, the conditional variance (1.6) vanishes, so x is an absorbing state and $\mathbf{A}_t$ is a zero operator.

Remark 1.6.2. *When $\sigma\tau = 0$, conditions (1.9) reduce to the assumption that $-1 \leqslant q \leqslant 1$. In this case, we have $\xi = 1 + q + |1 - q| = 2$ and*

$$\xi_0 = \tau + (1 - q)t + \sigma t^2, \qquad \xi_1 = \sigma t, \qquad \xi_2 = \theta - \eta t, \qquad \widetilde{q} = q.$$

Let us now consider the example of the Wiener process again and see what the theorem implies in this case.

Example 1.6.3. *Recall that the Wiener process is a quadratic harness $QH(0, 0; 0, 0; 1)$. Hence, we have $\xi_0 = 0$, and the formulas for $\widetilde{a}_n$ and $\widetilde{b}_n$ simplify to*

$$\widetilde{a}_n = x \qquad \text{and} \qquad \widetilde{b}_n = 0, \qquad n \in \mathbb{N}_0.$$

Therefore, the three-step recurrence takes the following form:

$$\begin{aligned} &\widetilde{W}_{-1}(y; x, t) = 0, \qquad \widetilde{W}_0(y; x, t) = 1, \\ &y\widetilde{W}_n(y; x, t) = \widetilde{W}_{n+1}(y; x, t) + x\widetilde{W}_n(y; x, t), \quad n \in \mathbb{N}_0. \end{aligned}$$

Consequently, $\widetilde{W}_n(y; x, t) = (y-x)^n$, $n \in \mathbb{N}_0$, and the polynomial sequence $\{\widetilde{W}_n(y; x, t)\}_{n=0}^{\infty}$ is orthogonal with respect to the Dirac measure at x. Indeed, it is easy to see that

$$\int_{\mathbb{R}} \widetilde{W}_n(y; x, t)\widetilde{W}_k(y; x, t)\delta_x(\mathrm{d}y) = \int_{\mathbb{R}} (y - x)^{n+k}\delta_x(\mathrm{d}y) = \mathbb{1}_{\{n=k=0\}}.$$

Therefore, in the case of the Wiener process, Theorem 1.6.1 provides the results that are consistent with those presented in Example 1.5.1.

1.7. Organization of the thesis

The thesis is organized as follows.

In the next chapter, we will provide an overview of the algebra $\mathcal{Q}$ of all infinite sequences of polynomials and give the necessary tools for the subsequent analysis in this chapter. Then, we will present the proof of Theorem 1.6.1, which we will divide into two parts. In the first part, we are going to show that the case $\sigma\tau > 0$ can be reduced to the case $\tau = 0$. In the second part, we will give a proof for the remaining case $\sigma\tau = 0$ under additional assumptions (Assumptions A1–A3). The reason for using Assumptions A1–A3 here is that they make the proof of Theorem 1.6.1 much easier; we will put a great deal of effort later to show that these assumptions are, in fact, implied by (1.9) along with $\sigma\tau = 0$.

Chapter 3 is devoted to extending the domain of the infinitesimal generator by including a class of bounded continuous functions with bounded continuous second derivatives. To do this, we will be using weak convergence of certain measures to the measure $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$, appearing in Theorem 1.6.1.

As it was mentioned earlier, extensive additional work is required to prove that Assumptions A1–A3 are implied by (1.9) combined with $\sigma\tau = 0$. This work is done in Chapters 4–6.

Chapters 4 and 5 present some additional results on the algebra $\mathcal{Q}$ that were not covered in Chapter 2. In particular, we introduce there certain linear subspaces of $\mathcal{Q}$, define some linear operators on $\mathcal{Q}$, and study properties of specially chosen elements of $\mathcal{Q}$.

The results on $\mathcal{Q}$ introduced in Chapters 4 and 5 are crucial for the task of proving that Assumptions A1–A3 follow from (1.9) and $\sigma\tau = 0$, what is the content of Chapter 6.

In Chapters 7–9 we study the infinitesimal generators of quadratic harnesses in special cases: $QH(\eta, \theta; \sigma, \tau; -\sigma\tau)$, $QH(\eta, \theta; \sigma, \tau; -1)$, $QH(\eta, \theta; \sigma, \tau; 1 - 2\sqrt{\sigma\tau})$. We will analyze their properties, and verify that the relevant results obtained in Chapters 2 and 3 coincide with those known from the literature.

Chapter 10 is a short summary with a discussion of the conclusions drawn from the results presented in the thesis.

In the appendix, we recall some basic results from the theory of orthogonal polynomials. We also clarify and unify certain terminological issues resulting from various approaches found in the literature.

Chapter 2

Proof of Theorem 1.6.1

In this chapter, we present a proof of the main theorem of the thesis. The presented proof works under additional assumptions (Assumptions A1–A3). This allows us to highlight the main steps of the proof and, hopefully, enhance its readability. A very technical task of replacing Assumptions A1–A3 with assumptions (1.9) appearing in Theorem 1.6.1 is postponed to Chapters 4–6.

We start by introducing the algebra $\mathcal{Q}$ of infinite sequences of polynomials, which plays a crucial role in our analysis.

2.1. Algebra of infinite sequences of polynomials

The algebra $\mathcal{Q}$ of all infinite sequences of polynomials was introduced in [24] to study the properties of polynomial processes. It is defined as a linear space of all infinite sequences of polynomials in a real variable x, with a non-commutative multiplication $\mathbb{R} = \mathbb{P}\mathbb{Q}$ for $\mathbb{P} = (P_0, P_1, \ldots)$, $\mathbb{Q} = (Q_0, Q_1, \ldots)$, and $\mathbb{R} = (R_0, R_1, \ldots) \in \mathcal{Q}$ given by

$$R_k(x) = \sum_{j=0}^{\deg(Q_k)} [Q_k]_j P_j(x), \quad k \geqslant 0, \tag{2.1}$$

where $[Q_k]_j$ denotes the coefficient at x^j in the polynomial Q_k. The element

$$\mathbb{E} = (1, x, x^2, x^3, \ldots) \tag{2.2}$$

is then the identity in $\mathcal{Q}$. Furthermore, if $\deg P_n = n$ for all $n \in \mathbb{N}_0$, then $\mathbb{P} = (P_0, P_1, \ldots)$ is invertible in $\mathcal{Q}$, see e.g. Proposition 1.2 in [24].

We single out two elements of $\mathcal{Q}$ that will play a fundamental role in the subsequent analysis:

$$\mathbb{D} := (0, 1, x, x^2, \ldots) \qquad \text{and} \qquad \mathbb{F} := (x, x^2, x^3, \ldots). \tag{2.3}$$

It can be easily verified that

$$\mathbb{D}\mathbb{F} = \mathbb{E}, \tag{2.4}$$

but $\mathbb{E} - \mathbb{F}\mathbb{D} = (1, 0, 0, \ldots)$, so $\mathbb{D}$ and $\mathbb{F}$ do not commute. Moreover, for any element $\mathbb{P} = (P_0, P_1, \ldots) \in \mathcal{Q}$ we have

$$\mathbb{P}(\mathbb{E} - \mathbb{F}\mathbb{D}) = P_0(x)(\mathbb{E} - \mathbb{F}\mathbb{D}). \tag{2.5}$$

Another key element under study is defined as follows:

$$\mathbb{D}_q := \sum_{k=0}^{\infty} q^k \mathbb{F}^k \mathbb{D}^{k+1}. \tag{2.6}$$

Since $\mathbb{D}^k$ has zeros in the first k coordinates, the series in the above expression consists of finite sums in each coordinate. Consequently, $\mathbb{D}_q$ is well-defined as an element of $\mathcal{Q}$. Moreover, in terms of the q-notation, recall (1.20), we can express it as:

$$\mathbb{D}_q = (0, [1]_q, [2]_q x, [3]_q x^2, \ldots). \tag{2.7}$$

In particular,

$$\mathbb{D}_1 = (0, 1, 2x, 3x^2, \ldots) \tag{2.8}$$

represents the classical derivative. Furthermore, in view of (2.5),

$$\mathbb{D}_q(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{O}. \tag{2.9}$$

By $\mathbb{O}$ we will denote the element of $\mathcal{Q}$ with all entries equal to zero, i.e., $\mathbb{O} := (0, 0, 0, \ldots)$. If $\mathbb{X} \in \mathcal{Q}$ additionally depends on a parameter $z \in \mathbb{R}$, we will write $\mathbb{X}(z)$. It should be understood that each coordinate of $\mathbb{X}(z)$ is a polynomial in the variable x with coefficients that depend on z.

In the thesis, we will need to evaluate the product $\mathbb{X}(z)\mathbb{Y}(z)$ at $z := x$. However, this operation requires caution, as even if $\mathbb{X}(z)|_{z:=x}, \mathbb{Y}(z)|_{z:=x} \in \mathcal{Q}$, the equality $(\mathbb{X}(z)\mathbb{Y}(z))|_{z:=x} = \mathbb{X}(z)|_{z:=x}\,\mathbb{Y}(z)|_{z:=x}$ may not hold, as shown in the following example.

Example 2.1.1. *Let* $\mathbb{X}(z) = \mathbb{Y}(z) = z(\mathbb{E} - \mathbb{F}\mathbb{D})$. *Then* $\mathbb{X}(z)|_{z:=x} = \mathbb{F}(\mathbb{E} - \mathbb{F}\mathbb{D}) \in \mathcal{Q}$, *and* $\mathbb{X}(z)|_{z:=x}\,\mathbb{Y}(z)|_{z:=x} = \mathbb{0}$ *by* (2.4). *However,*

$$(\mathbb{X}(z)\mathbb{Y}(z))|_{z:=x} = (z^2(\mathbb{E} - \mathbb{F}\mathbb{D}))|_{z:=x} = \mathbb{F}^2(\mathbb{E} - \mathbb{F}\mathbb{D}).$$

On the other hand, the following identity holds:

$$\big(\mathbb{X}(z)|_{z:=x}\mathbb{Y}(z)\big)|_{z:=x} = (z\mathbb{F}(\mathbb{E} - \mathbb{F}\mathbb{D}))|_{z:=x} = \mathbb{F}^2(\mathbb{E} - \mathbb{F}\mathbb{D}).$$

It is not a coincidence that the last two expressions in the above example are equal.

Remark 2.1.2. *Let* $\mathbb{X}(z), \mathbb{Y}(z) \in \mathcal{Q}$ *for all* $z \in \mathbb{R}$. *Suppose that all coordinates of* $\mathbb{X}(z)$ *are polynomials in* z. *Then* $\mathbb{X}(z)|_{z:=x} \in \mathcal{Q}$, *and*

$$(\mathbb{X}(z)\mathbb{Y}(z))|_{z:=x} = \big(\mathbb{X}(z)|_{z:=x}\mathbb{Y}(z)\big)|_{z:=x}.$$

Proof. Given the assumption that all coordinates of $\mathbb{X}(z)$ are polynomials in z, we see that the nth coordinate of $\mathbb{X}(z)$ evaluated at $z := x$ is a polynomial in x, i.e., $\mathbb{X}(z)|_{z:=x} \in \mathcal{Q}$. Note that $X_n(z)$ and $Y_n(z)$, the nth coordinates of $\mathbb{X}(z)$ and $\mathbb{Y}(z)$, respectively, can be written as

$$X_n(z) = \sum_{k=0}^{M_n} a_{k,n}(z)x^k \qquad \text{and} \qquad Y_n(z) = \sum_{k=0}^{N_n} b_{k,n}(z)x^k$$

for some $M_n, N_n \in \mathbb{N}_0$ and some coefficients $\{a_{k,n}(z)\}_{k=0}^{M_n} \in \mathbb{R}^{M_n+1}$, $\{b_{k,n}(z)\}_{k=0}^{N_n} \in \mathbb{R}^{N_n+1}$, $n \in \mathbb{N}_0$. Then, by (2.1), the nth coordinate of $\mathbb{X}(z)\mathbb{Y}(z)$ is equal to

$$\sum_{k=0}^{N_n} b_{k,n}(z) \sum_{m=0}^{M_k} a_{m,k}(z)x^m,$$

while the nth coordinate of $\mathbb{X}(z)|_{z:=x}\mathbb{Y}(z)$ is equal to

$$\sum_{k=0}^{N_n} b_{k,n}(z) \sum_{m=0}^{M_k} a_{m,k}(x)x^m.$$

Therefore, by looking at these two objects as functions of z and inserting $z := x$, we obtain the desired equality coordinate-wise. $\square$

2.2. Infinitesimal generator as an element of the algebra $\mathcal{Q}$

As mentioned in the introduction, the infinitesimal generator $\mathbf{A}_t$ of the quadratic harness $QH(\eta, \theta; \sigma, \tau; q)$ when acting on a polynomial also gives a polynomial. As a result, we can represent $\mathbf{A}_t$, $t \geqslant 0$, as an element $\mathbb{A}_t \in \mathcal{Q}$ with the nth coordinate given by $\mathbf{A}_t(x^n)$ for $n \in \mathbb{N}_0$, i.e.,

$$\mathbb{A}_t = \left(\mathbf{A}_t(1), \mathbf{A}_t(x), \mathbf{A}_t(x^2), \ldots\right),$$

see [24, Section 1.4]. Furthermore, it is obvious how to recover the formula for $\mathbf{A}_t$ acting on polynomials from the element $\mathbb{A}_t$ (by using the linearity of $\mathbf{A}_t$).
As [24] shows, using the language of the abstract algebra can be beneficial as it makes it easier to encode and represent some properties of quadratic harnesses. Also, algebraic formulations make proofs of many facts simpler and more universal. Moreover, it seems difficult to directly observe some of the relations proved in [24] without employing the algebraic framework.
In particular, it turns out that

$$\mathbb{H}_t := \mathbb{A}_t\mathbb{F} - \mathbb{F}\mathbb{A}_t, \tag{2.10}$$

which will be called a *pre-generator*, satisfies the *q-commutation equation*:

$$(1+\sigma t)\mathbb{H}_t\mathbb{F} - (q-\sigma t)\mathbb{F}\mathbb{H}_t = \mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2 + (\theta - \eta t)\mathbb{H}_t + (\tau + (1-q)t + \sigma t^2)\mathbb{H}_t^2, \qquad t \geqslant 0, \tag{2.11}$$

with the initial condition

$$\mathbb{H}_t(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}, \tag{2.12}$$

compare with [24, Theorem 2.1]. Proposition 2.4 in [24] states that if $\sigma\tau \neq 1$, then $\mathbb{H}_t$ is the unique solution of the q-commutation equation (2.11) with the initial condition (2.12).
From this fact, we can recover $\mathbb{A}_t$ using the formula:

$$\mathbb{A}_t = \sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{H}_t \mathbb{D}^{k+1}, \tag{2.13}$$

see [24, (3.8)]. The series (2.13) is well-defined since $\mathbb{D}^{k+1}$ has zeros on the first $k+1$ coordinates (and consequently, the series has a finite number of nonzero summands coordinate-wise).

Direct, purely algebraic solutions of the q-commutation equation have been found in only two cases: free quadratic harnesses $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$ and the classical version of quantum Bessel processes $QH(\eta,\theta;0,0;1)$, which are discussed in Chapters 7 and 9, respectively. A solution is also known when $\sigma=0$, see [44]. However, this one does not follow from the q-commutation equation only; instead, it requires an additional object—an auxiliary family of certain orthogonal polynomials.

The goal of this thesis is to show that the approach presented in [44], combining the q-commutation equation with some supplemental orthogonal polynomials, can be applied in general. However, the extension is far from being straightforward. Therefore, in this chapter, we provide an outline of the main ideas used to prove Theorem 1.6.1. This will also motivate the introduction of various auxiliary tools and highlight some useful identities involving special elements of the algebra $\mathcal{Q}$.

We consider two cases separately: $\sigma\tau>0$ and $\sigma\tau=0$. Firstly, we will show that the case $\sigma\tau>0$ can be reduced to the case $\tau=0$. Secondly, we will provide the proof when $\sigma\tau=0$ under additional assumptions (Assumptions A1–A3).

2.2.1. Algebraic infinitesimal generator when $\sigma\tau>0$.

Let us consider the pre-generator $\mathbb{H}_t$ of $QH(\eta,\theta;\sigma,\tau;q)$ with the parameters satisfying $\sigma\tau>0$ and (1.9). Under these conditions, $\mathbb{H}_t$ is uniquely determined by (2.11) and (2.12) since $\sigma\tau\neq 1$. We will show that the case $\sigma\tau>0$ can be reduced to the case $\tau=0$.

Proposition 2.2.1. *Assume $\sigma\tau>0$ and (1.9). Then the infinitesimal generator $\mathbb{A}_t$ of $QH(\eta,\theta;\sigma,\tau;q)$ at time $t\geqslant 0$ is given by*

$$\mathbb{A}_t=\tfrac{2}{\xi}\widehat{\mathbb{A}}_{\widetilde{t}},\tag{2.14}$$

where $\widehat{\mathbb{A}}_{\widetilde{t}}\in\mathcal{Q}$ is an infinitesimal generator of $QH(\eta,\widetilde{\theta};\sigma,0;\widetilde{q})$ at time $\widetilde{t}\geqslant 0$.
The expressions for ξ and $\widetilde{q}$ can be found in (1.21) *and* (1.24)*, respectively. Additionally,*

$\widetilde{\theta}$ along with $\widetilde{t}$ are defined as:

$$\widetilde{\theta} := \tfrac{2\theta}{\xi} + \tfrac{4\eta\tau}{\xi\left(1-q+\sqrt{(1-q)^2-4\sigma\tau}\right)} \tag{2.15}$$

and

$$\widetilde{t} := \tfrac{4(\tau+(1-q)t+\sigma t^2)}{\xi\left(1-q+2\sigma t+\sqrt{(1-q)^2-4\sigma\tau}\right)}. \tag{2.16}$$

We will prove an auxiliary lemma that ensures that the parameters mentioned in the statement of Proposition 2.2.1 are well-defined under the given assumptions.

Lemma 2.2.2. *Assuming $\sigma\tau > 0$ and (1.9), the parameters $\widetilde{\theta}$, $\widetilde{t}$, and $\widetilde{q}$ are well-defined and satisfy*

$$\widetilde{t} \geqslant 0 \qquad \text{and} \qquad \widetilde{q} \in [-1, 1].$$

Proof. As explained in the introduction, ξ given in (1.21) is positive. Furthermore, $1 - q + \sqrt{(1-q)^2 - 4\sigma\tau} > 0$ since

$$1 - q \geqslant 2\sqrt{\sigma\tau} > 0. \tag{2.17}$$

Moreover, $1-q+2\sigma t+\sqrt{(1-q)^2 - 4\sigma\tau} > 0$ because $2\sigma t \geqslant 0$. As a result, all denominators used in the expressions for $\widetilde{q}$, $\widetilde{\theta}$, and $\widetilde{t}$ are nonzero. Thus $\widetilde{q}$, $\widetilde{\theta}$ and $\widetilde{t}$ are well defined.
Let us proceed to the second part of the proof. Note that $\widetilde{t} \geqslant 0$ since $t \geqslant 0$, $1 - q > 0$ and $\sigma, \tau \geqslant 0$, see (1.5).
Moreover, (2.17) yields that $(1-q)^2 \geqslant 4\sigma\tau$, hence

$$(1-q)^2 - 4\sigma\tau + (1+q)\sqrt{(1-q)^2 - 4\sigma\tau} \geqslant 0.$$

The above implies that $\widetilde{q} \leqslant 1$. On the other hand, the condition $\widetilde{q} \geqslant -1$ is equivalent to

$$(1+q)^2 + (1+q)\sqrt{(1-q)^2 - 4\sigma\tau} = (1+q)\xi \geqslant 0,$$

which follows from (1.9). $\square$

Since all parameters are well-defined, the remaining task is to prove the formula (2.14).

Proof of Proposition 2.2.1. Direct calculations show that:

$$\begin{aligned} &\tfrac{\xi}{2}(1+\sigma\widetilde{t}) = 1+\sigma t, && \tfrac{\xi}{2}(\widetilde{q}-\sigma\widetilde{t}) = q-\sigma t, \\ &\tfrac{\xi}{2}(\widetilde{\theta}-\eta\widetilde{t}) = \theta-\eta t, && \tfrac{\xi^2}{4}(1-\widetilde{q}+\sigma\widetilde{t})\widetilde{t} = \tau+(1-q)t+\sigma t^2. \end{aligned} \tag{2.18}$$

Define $\mathbb{G} := \frac{\xi}{2}\mathbb{H}_t$. In terms of $\mathbb{G}$, the q-commutation equation (2.11) takes the form:

$$(1+\sigma\widetilde{t})\mathbb{G}\mathbb{F} - (\widetilde{q}-\sigma\widetilde{t})\mathbb{F}\mathbb{G} = \mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2 + (\widetilde{\theta}-\eta\widetilde{t})\mathbb{G} + (1-\widetilde{q}+\sigma\widetilde{t})\widetilde{t}\,\mathbb{G}^2$$

with the initial condition $\mathbb{G}(\mathbb{E}-\mathbb{F}\mathbb{D}) = \mathbb{0}$. Therefore, from the uniqueness of the solution of the q-commutation equation, $\mathbb{G}$ is the pre-generator of $QH(\eta,\widetilde{\theta};\sigma,0;\widetilde{q})$ at time $\widetilde{t} \geqslant 0$. In view of (2.13), we get the desired result. □

Summing up, we have shown that it is sufficient to find the solution to the q-commutation equation only in the case when $\sigma\tau = 0$ and $q \in [-1,1]$. By choosing the parameters appropriately and applying the time scaling (2.16), we can then obtain the formula for the pre-generator (or equivalently the infinitesimal generator) in the general case.

2.2.2. Algebraic infinitesimal generator when $\sigma\tau = 0$.

In this case, we prove the result under additional assumptions (Assumptions A1–A3). To formulate these assumptions, we need to introduce some additional objects. According to Remark 1.6.2, the parameters ξ_0 and ξ_2 simplify considerably when $\sigma\tau = 0$. To avoid confusion and the need for frequent referencing to the assumption $\sigma\tau = 0$, we will introduce new parameters:

$$\kappa_0 := \tau + (1-q)t + \sigma t^2 \qquad \text{and} \qquad \kappa_2 := \theta - \eta t, \tag{2.19}$$

which are versions of ξ_0 and ξ_2 with $\sigma\tau = 0$ applied. Similarly, by $\{W_n(\cdot;z,t)\}_{n=0}^{\infty}$ we will mean the polynomials $\{\widetilde{W}_n(\cdot;z,t)\}_{n=0}^{\infty}$ when $\sigma\tau = 0$. Thus, the three-step recurrence for $\{W_n(\cdot;z,t)\}_{n=0}^{\infty}$ takes the form:

$$\begin{aligned} &W_{-1}(x;z,t) = 0, \qquad W_0(x;z,t) = 1, \\ &xW_n(x;z,t) = W_{n+1}(x;z,t) + a_n(z)W_n(x;z,t) + b_n(z)W_{n-1}(x;z,t), \quad n \geqslant 0, \end{aligned} \tag{2.20}$$

with

$$a_n(z) := \kappa_0 \tfrac{[n]_q[n+1]_q}{1+\sigma t[2n+1]_q} \left[\sigma(\gamma_{n+1}(z) + \gamma_n(z)) + \eta\right] + \gamma_{n+1}(z),$$
$$b_n(z) := \kappa_0 \tfrac{[n]_q[n+1]_q(1+\sigma t[n-1]_q)(1+\sigma t[n]_q)}{(1+\sigma t[2n-1]_q)(1+\sigma t[2n+1]_q)} \left[1 + \gamma_n(z)(\sigma\gamma_n(z) + \eta)\right]$$

and

$$\gamma_n(z) := \tfrac{zq^n + \eta\kappa_0[n]_q^2 + \kappa_2[n]_q(1+\sigma t[n]_q)}{1+\sigma t[2n]_q}. \tag{2.21}$$

Instead of the original q-commutation equation (2.11), we consider a similar version given by:

$$(1+\sigma t)\widetilde{\mathbb{H}}_t\mathbb{F} - (q-\sigma t)\mathbb{F}\widetilde{\mathbb{H}}_t = \mathbb{E} + \kappa_2\widetilde{\mathbb{H}}_t + \kappa_0\widetilde{\mathbb{H}}_t\mathcal{R}(\mathbb{E})\widetilde{\mathbb{H}}_t \tag{2.22}$$

with the initial condition

$$\widetilde{\mathbb{H}}_t(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}, \tag{2.23}$$

where $\mathcal{R} : \mathcal{Q} \to \mathcal{Q}$ acts on any element $\mathbb{X} \in \mathcal{Q}$ as follows:

$$\mathcal{R}(\mathbb{X}) := \mathbb{E} + \eta\mathbb{X}\mathbb{F} + \sigma(\mathbb{X}\mathbb{F})^2. \tag{2.24}$$

If $\widetilde{\mathbb{H}}_t$ satisfies (2.22) and (2.23), then $\mathbb{H}_t = \mathcal{R}(\mathbb{E})\widetilde{\mathbb{H}}_t$ satisfies (2.11) and (2.12). Consequently, such $\mathbb{H}_t$ is the pre-generator of $QH(\eta,\theta;\sigma,\tau;q)$ due to the uniqueness of the solution of the q-commutation equation, as stated in Proposition 2.4 in [24].
Therefore, we will focus on solving (2.22). We will associate $\widetilde{\mathbb{H}}_t$ with the polynomials $\{W_n(\cdot; z,t)\}_{n=0}^{\infty}$ in the variable x given by (2.20). In the algebra $\mathcal{Q}$, we represent these polynomials as $\mathbb{W}(z,t)$:

$$\mathbb{W}(z,t) := \left(W_0(x;z,t), W_1(x;z,t), W_2(x;z,t), \dots\right). \tag{2.25}$$

Every polynomial W_n is a monic polynomial of degree n in the variable x, so [24, Proposition 1.2] implies that $\mathbb{W}(z,t)$ is invertible (with the inverse denoted as $\mathbb{W}(z,t)^{-1}$).
From the definition of multiplication in $\mathcal{Q}$, we can express the three-step recurrence (2.20) algebraically as follows:

$$\mathbb{F}\mathbb{W}(z,t)\mathbb{D} + \mathbb{E} - \mathbb{F}\mathbb{D} = \mathbb{W}(z,t)\mathbb{S}(z,t), \tag{2.26}$$

where $\mathbb{S}(z,t) \in \mathcal{Q}$ with its nth coordinate given by

$$x^n + a_{n-1}(z)x^{n-1} + b_{n-1}(z)x^{n-2}, \qquad n \in \mathbb{N}_0. \tag{2.27}$$

According to the assumed q-notation convention (recall (1.20)), the above expression should be interpreted as 1 when $n = 0$, and $x + a_0(z)$ when $n = 1$. Consequently,

$$\mathbb{S}(z,t)(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D}$$

by (2.5). As a result, in view of (2.4), multiplying (2.26) from the right by $\mathbb{E} - \mathbb{F}\mathbb{D}$ and $\mathbb{F}$, respectively, leads to the following equations:

$$\mathbb{W}(z,t)(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D} \tag{2.28}$$

and

$$\mathbb{F}\mathbb{W}(z,t) = \mathbb{W}(z,t)\mathbb{S}(z,t)\mathbb{F}. \tag{2.29}$$

After collecting all the necessary identities for $\mathbb{W}(z,t)$, we can return to the question of finding a solution of (2.22) satisfying the initial condition (2.23). Let

$$\mathbb{Z} := (\mathbb{E} + \sigma t\mathbb{F}\mathbb{D}_q)\mathbb{D}_q. \tag{2.30}$$

Below we present Assumptions A1–A3 that have been mentioned at the beginning of Chapter 2.

Assumption A1:
For all $z \in \mathbb{R}$ and $t \geqslant 0$, there exist invertible elements $\mathbb{U}(z,t), \mathbb{Y}(z,t) \in \mathcal{Q}$ satisfying

$$\mathbb{D}_q\mathbb{Y}(z,t) = \mathbb{U}(z,t)\mathbb{Z}.$$

Assumption A2:
For every $z \in \mathbb{R}$, $t \geqslant 0$, and for $\mathbb{U}(z,t)$ and $\mathbb{Y}(z,t)$ from Assumption A1, we have

$$\begin{aligned}(1+\sigma t)\mathbb{D}_q\mathbb{S}(z,t)\mathbb{F}\mathbb{Y}(z,t) - (q-\sigma t)\mathbb{U}(z,t)\mathbb{S}(z,t)\mathbb{F}\mathbb{Z}&\\ = \mathbb{U}(z,t)\mathbb{Y}(z,t) + \kappa_2\mathbb{D}_q\mathbb{Y}(z,t) + \kappa_0\mathbb{D}_q\mathcal{R}(\mathbb{S}(z,t))\mathbb{Z}&.\end{aligned}$$

Assumption A3:
For all $z \in \mathbb{R}$ and $t \geqslant 0$, and for $\mathbb{U}(z,t)$ from Assumption A1, there exists $\widetilde{\mathbb{U}}(z,t) \in \mathcal{Q}$ such that

$$\mathbb{U}(z,t)^{-1}\mathbb{D}_q\mathbb{S}(z,t)\mathbb{F} - \mathbb{S}(z,t)\mathbb{F}\mathbb{U}(z,t)^{-1}\mathbb{D}_q = (\mathbb{S}(z,t)\mathbb{F} - z\mathbb{E})\widetilde{\mathbb{U}}(z,t) + \tfrac{1}{1+\sigma t}(\mathbb{E} - \mathbb{F}\mathbb{D}).$$

Throughout this chapter, we will proceed under these assumptions.

Proposition 2.2.3. *Given Assumptions A1–A3, for all $z \in \mathbb{R}$ and $t \geqslant 0$, we have:*

$$\widetilde{\mathbb{H}}_t = \mathbb{W}(z,t)\mathbb{U}(z,t)^{-1}\mathbb{D}_q\mathbb{W}(z,t)^{-1}, \tag{2.31}$$

$$\widetilde{\mathbb{M}}_t := \widetilde{\mathbb{H}}_t\mathbb{F} - \mathbb{F}\widetilde{\mathbb{H}}_t = (\mathbb{F} - z\mathbb{E})\mathbb{W}(z,t)\widetilde{\mathbb{U}}(z,t)\mathbb{W}(z,t)^{-1} + \tfrac{1}{1+\sigma t}(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{W}(z,t)^{-1}. \tag{2.32}$$

Proof. Fix $z \in \mathbb{R}$ and $t \geqslant 0$. From now on, we suppress (z,t) as arguments of functions with values in $\mathcal{Q}$.
To prove (2.31), we need to show that $\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}$ satisfies both (2.23) and (2.22). Let us begin with the initial condition. Using (2.28) and (2.9), we have

$$\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0},$$

so (2.23) holds.
Next, we will show that (2.22) is also satisfied. Multiplying the formula from Assumption A2 from the left by $\mathbb{W}\mathbb{U}^{-1}$ and from the right by $\mathbb{Y}^{-1}\mathbb{W}^{-1}$, and using Assumption A1, we obtain

$$\begin{aligned}(1+\sigma t)\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{W}^{-1} - &(q-\sigma t)\mathbb{W}\mathbb{S}\mathbb{F}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}\\ &= \mathbb{E} + \kappa_2\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1} + \kappa_0\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathcal{R}(\mathbb{S})\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}.\end{aligned}$$

According to (2.29), we have $\mathcal{R}(\mathbb{E})\mathbb{W} = \mathbb{W}\mathcal{R}(\mathbb{S})$. Hence we can rewrite the above equation as

$$\begin{aligned}(1+\sigma t)\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}\mathbb{F} - (q-\sigma t)\mathbb{F}\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}&\\ = \mathbb{E} + \kappa_2\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1} + \kappa_0\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}\mathcal{R}(\mathbb{E})\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}.&\end{aligned}$$

This equation is exactly (2.22) with $\widetilde{\mathbb{H}}_t$ replaced by $\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}$. Therefore, we have proven that (2.31) holds.

To prove (2.32), we multiply the formula in Assumption A3 from the left by $\mathbb{W}$ and from the right by $\mathbb{W}^{-1}$ to obtain

$$\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{W}^{-1} - \mathbb{W}\mathbb{S}\mathbb{F}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1} = \mathbb{W}(\mathbb{S}\mathbb{F} - z\mathbb{E})\widetilde{\mathbb{U}}\mathbb{W}^{-1} + \tfrac{1}{1+\sigma t}\mathbb{W}(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{W}^{-1}.$$

Formulas (2.29) and (2.28) lead to

$$\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1}\mathbb{F} - \mathbb{F}\mathbb{W}\mathbb{U}^{-1}\mathbb{D}_q\mathbb{W}^{-1} = (\mathbb{F} - z\mathbb{E})\mathbb{W}\widetilde{\mathbb{U}}\mathbb{W}^{-1} + \tfrac{1}{1+\sigma t}(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{W}^{-1}.$$

Finally, by substituting (2.31) into the above equation, we obtain (2.32). This completes the proof. □

Thus, we have found the solution $\widetilde{\mathbb{H}}_t$ of (2.22) satisfying (2.23). However, we have done so under the additional assumptions. As it will be shown in Chapters 4–6, these assumptions are in fact implied by (1.9). Also in Chapters 5 and 6, some explicit formulas for $\mathbb{U}(z,t)$, $\mathbb{Y}(z,t)$, and $\widetilde{\mathbb{U}}(z,t)$ will be revealed.

The solution $\widetilde{\mathbb{H}}_t$ is given in terms of $\mathbb{W}(z,t)$ and $\mathbb{U}(z,t)$. However, due to complicated formulas for these elements, a direct deduction of Theorem 1.6.1 from the formula (2.31) seems to be challenging. Surprisingly, a more effective approach follows via the identity (2.32). A striking feature of this approach is that the explicit formula for $\widetilde{\mathbb{U}}(z,t)$ is irrelevant.

The most important term is $\mathbb{F} - z\mathbb{E}$, since $\mathbb{F} - z\mathbb{E}|_{z:=x} = \mathbb{0}$. In fact, note that the left-hand side of (2.32) does not depend on z, while the right-hand side does (z appears not only in the expression $\mathbb{F} - z\mathbb{E}$, but also in $\mathbb{W} = \mathbb{W}(z,t)$ and $\widetilde{\mathbb{U}} = \widetilde{\mathbb{U}}(z,t)$). Hence,

$$\widetilde{\mathbb{M}}_t = \widetilde{\mathbb{M}}_t|_{z:=x} = (\mathbb{F} - z\mathbb{E})\mathbb{W}(z,t)\widetilde{\mathbb{U}}(z,t)\mathbb{W}(z,t)^{-1}|_{z:=x} + \tfrac{1}{1+\sigma t}(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{W}(z,t)^{-1}|_{z:=x}.$$

Consequently, Remark 2.1.2 yields

$$\widetilde{\mathbb{M}}_t = \tfrac{1}{1+\sigma t}(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{W}(z,t)^{-1}|_{z:=x}. \tag{2.33}$$

From the initial condition for $\widetilde{\mathbb{H}}_t$, we obtain $\widetilde{\mathbb{H}}_t\mathbb{F}\mathbb{D} = \widetilde{\mathbb{H}}_t$. Therefore, we can easily recover $\widetilde{\mathbb{H}}_t$ from $\widetilde{\mathbb{M}}_t$ by iterating the equality $\widetilde{\mathbb{H}}_t = \widetilde{\mathbb{M}}_t\mathbb{D} + \mathbb{F}\widetilde{\mathbb{H}}_t\mathbb{D}$. This immediately leads to

$$\widetilde{\mathbb{H}}_t = \sum_{k=0}^{\infty} \mathbb{F}^k \widetilde{\mathbb{M}}_t \mathbb{D}^{k+1},$$

where, coordinate-wise, all sums have finite numbers of nonzero terms. Hence,

$$\mathbb{H}_t = \mathcal{R}(\mathbb{E})\widetilde{\mathbb{H}}_t = \mathcal{R}(\mathbb{E}) \sum_{k=0}^{\infty} \mathbb{F}^k \widetilde{\mathbb{M}}_t \mathbb{D}^{k+1} \tag{2.34}$$

is a solution of the q-commutation equation (2.11). By (2.13) we get a formula for $\mathbb{A}_t$ from (2.34).

However, the form of $\mathbb{A}_t$ stemming from (2.13) is not our ultimate aim. To complete the proof of Theorem 1.6.1, we need to derive an integro-differential representation of $\mathbb{A}_t$.

2.3. Integro-differential representation for infinitesimal generators

This section builds on the results from the previous section and completes our proof of Theorem 1.6.1 under Assumptions A1–A3. As before, the cases $\sigma\tau = 0$ and $\sigma\tau > 0$ will be considered separately.

Definitions of a weakly orthogonal polynomial sequence and a moment functional, which will be frequently used in this section, are provided in the appendix.

2.3.1. The final part of the proof of Theorem 1.6.1 when $\sigma\tau = 0$.

Fix $t \geqslant 0$ and $z \in \mathbb{R}$. By generalized Favard's theorem, see Theorem A.0.1, the polynomials $\{W_n(x; z, t)\}_{n=0}^{\infty}$ given in (2.20) are weakly orthogonal (see Definition A.0.3) with respect to a moment functional $\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}$, which acts on polynomials in a variable $y \in \mathbb{R}$, i.e.,

$$\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[W_n(y; z, t)W_k(y; z, t)] = \chi_n \mathbb{1}(n = k), \qquad n, k \in \mathbb{N}_0,$$

where $\chi_0 \neq 0$. Without any loss of generality, we assume that the moment functional $\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}$ is normalized, i.e., $\chi_0 = 1$. When $n = 0$, the above formula becomes:

$$\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[W_k(y;z,t)] = \mathbb{1}(k=0), \qquad k \in \mathbb{N}_0. \tag{2.35}$$

To translate (2.35) into the language of the algebra $\mathcal{Q}$, we introduce an element $\mathbb{E}_y := (1, y, y^2, \ldots)$, $y \in \mathbb{R}$ (so all coordinates are polynomials of degree zero in the generic variable x). Clearly, $\mathbb{E}_y$ is a well-defined element in $\mathcal{Q}$. Note that multiplication from the left by $\mathbb{E}_y$ is a change of the variable from x to y in the polynomials in each coordinate. Therefore, (2.35) can be written as

$$\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y \mathbb{W}(z,t)] = \mathbb{E} - \mathbb{F}\mathbb{D}, \tag{2.36}$$

where $\mathcal{L}_{z,t,\theta,\tau,\eta,q}$ on the left-hand side of (2.36) acts coordinate-wise on $\mathbb{E}_y\mathbb{W}(z,t)$ (recall that $\mathbb{W}(z,t)$ is given by (2.25)). It is important to emphasize that $\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}$ acts only on polynomials in the variable y, while in the algebra $\mathcal{Q}$ we consider polynomials in the generic variable x. This distinction between the two variables is crucial.

Now, define $\mathbb{J}(z,t)$ as follows:

$$\mathbb{J}(z,t) := \Big(\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[1], \mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[y], \mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[y^2], \ldots\Big) = \mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y].$$

Note that $\mathbb{J}(z,t)$ is a well-defined element of the algebra $\mathcal{Q}$ since all coordinates are constant with respect to x. It encodes all moments of the moment functional $\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}$. Using the linearity of $\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}$ and equation (2.36), we obtain:

$$\mathbb{J}(z,t)\mathbb{W}(z,t) = \mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y\mathbb{W}(z,t)] = \mathbb{E} - \mathbb{F}\mathbb{D}. \tag{2.37}$$

In the first equality above, we used the definition of multiplication (recall (2.1)) and the fact that $\mathbb{W}(z,t)$ does not depend on y. Equation (2.37) is equivalent to:

$$\mathbb{J}(z,t) = (\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{W}(z,t)^{-1}.$$

Since this equality holds for all fixed $z \in \mathbb{R}$, comparing it with (2.33) implies that $\mathbb{J}(z,t)|_{z:=x} \in \mathcal{Q}$ and

$$\widetilde{\mathbb{M}}_t = \tfrac{1}{1+\sigma t}\mathbb{J}(z,t)|_{z:=x} = \tfrac{1}{1+\sigma t}\mathcal{L}_{z,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y]|_{z:=x} = \tfrac{1}{1+\sigma t}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y].$$

After inserting this into (2.34), we get

$$\mathbb{H}_t = \tfrac{1}{1+\sigma t}\mathcal{R}(\mathbb{E})\sum_{k=0}^{\infty}\mathbb{F}^k\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y]\mathbb{D}^{k+1}.$$

Our goal is to simplify the above expression; however, this is a delicate task that requires some careful justification.

Remark 2.3.1. *Suppose that $\mathbb{X} \in \mathcal{Q}$ do not depend on $y \in \mathbb{R}$. In general, the equality*

$$\mathbb{X}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y] = \mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{X}\mathbb{E}_y] \tag{2.38}$$

does not hold. For instance, when $\mathbb{X} = \mathbb{D}$, we have

$$\mathbb{D}\mathbb{E}_y = \mathbb{0}, \tag{2.39}$$

since all coordinates of $\mathbb{E}_y$ do not depend on x. Hence $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{D}\mathbb{E}_y] = \mathbb{0}$, while $\mathbb{D}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y] \neq \mathbb{0}$ if only $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{E}_y]$ has some entries that are polynomials in the generic variable x of degree at least 1.
However, (2.38) holds true for $\mathbb{X} = \mathbb{F}^k$, $k \in \mathbb{N}_0$, as $\mathbb{F}^k$ only multiplies each coordinate by x^k, and $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$ is a linear operator acting solely on polynomials in the variable y.
On the other hand, for any $\mathbb{X}$, $\widetilde{\mathbb{X}} \in \mathcal{Q}$ that do not depend on y, we observe that

$$\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{X}\mathbb{E}_y]\widetilde{\mathbb{X}} = \mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{X}\mathbb{E}_y\widetilde{\mathbb{X}}].$$

The above equation follows from (2.1) and the fact that $\widetilde{\mathbb{X}}$ does not depend on y.

In view of Remark 2.3.1 and the linearity of $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$,

$$\mathbb{H}_t = \tfrac{1}{1+\sigma t}\mathcal{R}(\mathbb{E})\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{Q}_y],$$

where
$$\mathbb{Q}_y := \sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{E}_y \mathbb{D}^{k+1}.$$

Using (2.13) and arguments analogous to those presented in Remark 2.3.1, we can quickly justify that
$$\mathbb{A}_t = \tfrac{1}{1+\sigma t}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}\left[\sum_{j=0}^{\infty} \mathbb{F}^j \mathbb{Q}_y \mathbb{D}^{j+1}\right].$$

It turns out that the expression in the square brackets can be considerably simplified to the form $\mathbb{D}_1\mathbb{Q}_y$. This follows from the fact that the identities (2.6) for $q = 1$ and (2.39) yield

$$\begin{aligned}\mathbb{D}_1\mathbb{Q}_y &= \left(\sum_{j=0}^{\infty} \mathbb{F}^j \mathbb{D}^{j+1}\right)\left(\sum_{k=0}^{\infty} \mathbb{F}^k \mathbb{E}_y \mathbb{D}^{k+1}\right) = \sum_{j=0}^{\infty} \mathbb{F}^j \left(\sum_{k=j+1}^{\infty} \mathbb{F}^{k-j-1}\mathbb{E}_y \mathbb{D}^{k+1}\right)\\ &= \sum_{j=0}^{\infty} \mathbb{F}^j \left(\sum_{k=j+1}^{\infty} \mathbb{F}^{k-j-1}\mathbb{E}_y \mathbb{D}^{k-j}\right)\mathbb{D}^{j+1} = \sum_{j=0}^{\infty} \mathbb{F}^j \mathbb{Q}_y \mathbb{D}^{j+1}.\end{aligned}$$

Consequently,
$$\mathbb{A}_t = \tfrac{1}{1+\sigma t}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[\mathbb{D}_1\mathbb{Q}_y].$$

Since $\mathbb{E}_y\mathbb{D}^{k+1}$ has the nth coordinate equal to zero for $n \leqslant k$, and equal to y^{n-k-1} for $n \geqslant k+1$, the nth coordinate of $\mathbb{Q}_y$ has the form

$$y^{n-1} + y^{n-2}x + \ldots + yx^{n-2} + x^{n-1} = \tfrac{y^n - x^n}{y-x}, \qquad n \in \mathbb{N}.$$

The 0th coordinate is zero. Differentiating with respect to x leads to

$$y^{n-2} + \ldots + (n-2)yx^{n-3} + (n-1)x^{n-2} = \tfrac{\partial}{\partial x}\tfrac{y^n - x^n}{y-x};$$

the left-hand side is the nth coordinate of $\mathbb{D}_1\mathbb{Q}_y$ by (2.1) and (2.8). Therefore, the nth coordinate of $\mathbb{D}_1\mathbb{Q}_y$ is equal to $\frac{\partial}{\partial x}\frac{y^n-x^n}{y-x}$. Consequently, the nth coordinate of $\mathbb{A}_t$ takes the following form:
$$\tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}\left[\tfrac{\partial}{\partial x}\tfrac{y^n - x^n}{y-x}\right].$$

Recall that the nth coordinate of $\mathbb{A}_t$ is also equal to $\mathbf{A}_t(x^n)$. As a result, from the linearity of $\mathbf{A}_t$, we have that for any polynomial f

$$(\mathbf{A}_t f)(x) = \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}\left[\tfrac{\partial}{\partial x}\,\tfrac{f(y)-f(x)}{y-x}\right]. \tag{2.40}$$

This ends the proof of the first part of Theorem 1.6.1.

We will now proceed to show that the moment functional $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$ is non-negative when $1+\eta x+\sigma x^2>0$. Note that

$$\mathbb{A}_t\mathbb{F}^2 - 2\mathbb{F}\mathbb{A}_t\mathbb{F} + \mathbb{F}^2\mathbb{A}_t = \mathbb{H}_t\mathbb{F} - \mathbb{F}\mathbb{H}_t = \tfrac{1}{1+\sigma t}(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\widetilde{\mathbb{M}}_t.$$

Coordinate-wise, this can be expressed as

$$\begin{aligned}\lim_{h\to 0^+}\int_{\mathbb{R}} \tfrac{y^{n+2}-x^{n+2}}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) - 2x\lim_{h\to 0^+}\int_{\mathbb{R}} \tfrac{y^{n+1}-x^{n+1}}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)&\\ +x^2\lim_{h\to 0^+}\int_{\mathbb{R}} \tfrac{y^{n}-x^{n}}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) &= \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[y^n]\end{aligned}$$

for all $n\in\mathbb{N}_0$. The left-hand side can be simplified to $\lim\limits_{h\to 0^+}\int_{\mathbb{R}} y^n\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$. Consequently, from the linearity of $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$ it follows that for any non-negative polynomial f (i.e., $f(y)\geqslant 0$ for all $y\in\mathbb{R}$), we have

$$0 \leqslant \lim_{h\to 0^+}\int_{\mathbb{R}} f(y)\tfrac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[f]. \tag{2.41}$$

The inequality holds true because the integral under the limit is non-negative for all $h>0$. Since $\sigma\geqslant 0$ (see (1.5)), $t\geqslant 0$ and $1+\eta x+\sigma x^2>0$, formula (2.41) implies that $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$ is a non-negative definite moment functional. Remark A.0.2 (in the appendix) implies that the product of consecutive coefficients at W_{n-1} from the three-step recurrence (2.20) is non-negative. Therefore, by [22, Theorem A.1.], there exists a probability measure $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$ such that for all polynomials f we have

$$\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}[f] = \int_{\mathbb{R}} f(y)\ \nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y). \tag{2.42}$$

Putting together (2.40) and (2.42) finishes the second part of the proof of Theorem 1.6.1 when $\sigma\tau = 0$.

2.3.2. The final part of the proof of Theorem 1.6.1 when $\sigma\tau > 0$.

Let us fix $\eta, \theta \in \mathbb{R}$, $\sigma, \tau > 0$ and $q \in [-1, 1 - 2\sqrt{\sigma\tau}]$.

Recall that Lemma 2.2.2 states that $\widetilde{q}$ given by (1.24) belongs to $[-1, 1]$. Therefore, the previous subsection provides, in particular, a formula (see (2.40)) for the infinitesimal generator of $QH(\eta, \widetilde{\theta}; \sigma, 0; \widetilde{q})$ for any $t \geqslant 0$, where $\widetilde{\theta}$ is defined in (2.15); we will denote the infinitesimal generator for this particular choice of parameters by $\widehat{\mathbf{A}}_t$.
In particular, we can evaluate $\widehat{\mathbf{A}}_t$ at time $t = \widetilde{t}$ (recall that the rescaled version $\widetilde{t}$ of t is given in (2.16)). Then, Proposition 2.2.1, read coordinate-wise, implies that

$$\mathbf{A}_t(x^n) = \tfrac{2}{\xi}\widehat{\mathbf{A}}_{\widetilde{t}}(x^n), \qquad n \in \mathbb{N}_0,$$

where by $\mathbf{A}_t$ we mean the infinitesimal generator of $QH(\eta, \theta; \sigma, \tau; q)$. Theorem 1.6.1 applied to $QH(\eta, \widetilde{\theta}; \sigma, 0; \widetilde{q})$ gives

$$\begin{aligned}(\mathbf{A}_t f)(x) = \tfrac{2}{\xi}(\widehat{\mathbf{A}}_{\widetilde{t}} f)(x) &= \tfrac{2}{\xi}\tfrac{1+\eta x+\sigma x^2}{1+\sigma\widetilde{t}}\mathcal{L}_{x,\widetilde{t},\eta,\widetilde{\theta},\sigma,0,\widetilde{q}}\left[\tfrac{\partial}{\partial x}\tfrac{f(y)-f(x)}{y-x}\right] \\ &= \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\mathcal{L}_{x,\widetilde{t},\eta,\widetilde{\theta},\sigma,0,\widetilde{q}}\left[\tfrac{\partial}{\partial x}\tfrac{f(y)-f(x)}{y-x}\right].\end{aligned}$$

The third equality comes from the first identity in (2.18).
Therefore, by replacing $\mathcal{L}_{x,\widetilde{t},\eta,\widetilde{\theta},\sigma,0,\widetilde{q}}$ with $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$, we obtain the representation stated in Theorem 1.6.1.
Moreover, we observe that $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$ makes the polynomials $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$ given in (1.23) orthogonal since these polynomials are derived from the polynomials $\{W_n(\cdot; x, \widetilde{t})\}_{n=0}^{\infty}$ by inserting the appropriate parameters.
To obtain the expressions for the three-term recursion coefficients $\widetilde{a}_n$, $\widetilde{b}_n$, and $\widetilde{\gamma}_n$ appearing in Theorem 1.6.1, we once again apply (2.18) and use the notation introduced in (1.22).
If $1 + \eta x + \sigma x^2 > 0$ then there exists an integral representation of $\mathcal{L}_{x,\widetilde{t},\eta,\widetilde{\theta},\sigma,0,\widetilde{q}}$. Hence, $\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$ also has the integral representation with $\nu_{x,t,\eta,\theta,\sigma,\tau,q} = \nu_{x,\widetilde{t},\eta,\widetilde{\theta},\sigma,0,\widetilde{q}}$.

Chapter 3

Extension of the domain of the infinitesimal generator

The main objective is to include in the domain of the infinitesimal generator a class of bounded continuous functions with bounded continuous second derivatives. For this purpose, we assume that

$$1 + \eta x + \sigma x^2 > 0. \tag{3.1}$$

Under this condition, the polynomials $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$ are orthogonal with respect to a probability measure $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$.

The chapter is organized as follows:

1. Recall that $\{\mathbb{P}_{s,t}(x, \mathrm{d}y) : x \in \mathbb{R}, 0 \leqslant s < t\}$ denotes the transition probabilities of the considered quadratic harness. We will prove that all moments of the measures $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x, \mathrm{d}y)$ and $\frac{(y-x)^2}{h}\mathbb{P}_{t-h,t}(x, \mathrm{d}y)$ converge to the corresponding moments of $\frac{1+\eta x+\sigma x^2}{1+\sigma t}\nu_{x,t,\eta,\theta,\sigma,\tau,q}$.
2. We will discuss conditions under which $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$ is determined by its moments.
3. We will extend the domain of the infinitesimal generator for certain parameters of quadratic harnesses by including a class of bounded continuous functions with bounded continuous second derivatives.

The first two steps are necessary to establish the weak convergence of the measures $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x, \mathrm{d}y)$ and $\frac{(y-x)^2}{h}\mathbb{P}_{t-h,t}(x, \mathrm{d}y)$ to $\frac{1+\eta x+\sigma x^2}{1+\sigma t}\nu_{x,t,\eta,\theta,\sigma,\tau,q}$. Weak convergence in turn is crucial for extending the domain of the infinitesimal generator.

3.1. Moment convergence

We will start by showing that the respective moments converge.

Lemma 3.1.1. *For all $t \geqslant 0$ and x satisfying (3.1), all moments of the measure $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ converge to the corresponding moments of $\frac{1+\eta x+\sigma x^2}{1+\sigma t}\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y)$ as h approaches 0 from the right.*
Similarly, for $t>0$ all moments of $\frac{(y-x)^2}{h}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$ tend to the corresponding moments of $\frac{1+\eta x+\sigma x^2}{1+\sigma t}\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y)$ as h goes to 0 from the right.

Proof. Fix $t \geqslant 0$. The definition of the right infinitesimal generator, see (1.15), implies that

$$\lim_{h\to 0^+}\int_{\mathbb{R}} y^n \tfrac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \mathbf{A}_t(x^{n+2}) - 2x\mathbf{A}_t(x^{n+1}) + x^2\mathbf{A}_t(x^n).$$

Theorem 1.6.1 and the identity

$$\tfrac{\partial}{\partial x}\tfrac{y^{n+1}-x^{n+1}}{y-x} = \tfrac{\partial}{\partial x}\tfrac{(y-x)y^n+x(y^n-x^n)}{y-x} = x\tfrac{\partial}{\partial x}\tfrac{y^n-x^n}{y-x} + \tfrac{y^n-x^n}{y-x}, \qquad n\in\mathbb{N}_0,$$

yield

$$\begin{aligned}
&\lim_{h\to 0^+}\int_{\mathbb{R}} y^n \tfrac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)\\
&\quad= \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\int_{\mathbb{R}}\tfrac{y^{n+1}-x^{n+1}}{y-x}\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y) - \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\int_{\mathbb{R}} x\tfrac{y^n-x^n}{y-x}\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y)\\
&\quad= \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\int_{\mathbb{R}} y^n\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y), \qquad n\in\mathbb{N}_0.
\end{aligned}$$

Analogously, we can show the same for $\frac{(y-x)^2}{h}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$ when $t>0$. ☐

Observe that $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ is a nonnegative measure. From (1.6) we get for all $h>0$ that

$$\int_{\mathbb{R}}\tfrac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \tfrac{\mathbb{V}\mathrm{ar}(X_{t+h}|X_t=x)}{h} = \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}.$$

This, together with similar arguments for $\frac{(y-x)^2}{h}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$, justifies the following remark:

Remark 3.1.2. *Let x satisfy (3.1). Then*

$$\tfrac{(1+\sigma t)(y-x)^2}{h(1+\eta x+\sigma x^2)}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) \qquad \text{and} \qquad \tfrac{(1+\sigma(t-h))(y-x)^2}{h(1+\eta x+\sigma x^2)}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$$

are probability measures for all $h > 0$ (the first measure) and for h such that $t \geqslant h > 0$ (the second measure).

3.2. Moment determinacy and weak convergence

We say that a probability measure μ is *determined by its moments* if μ is the only probability measure that has the moments

$$\int_{\mathbb{R}} x^k \mu(\mathrm{d}y), \qquad k \in \mathbb{N}_0.$$

This problem of moment determinacy is known as the Hamburger moment problem, see [27, Section 2.6]. It is well-known that if μ has a bounded support, then it is determined by its moments.
Now we will quickly verify that $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$ is determined by its moments in the following cases:

⋇ $q \in [-1, 1 - 2\sqrt{\sigma\tau})$. Under this condition, we have

$$0 < \sqrt{(1-q)^2 - 4\sigma\tau}(1 + q + \sqrt{(1-q)^2 - 4\sigma\tau}),$$

so $\widetilde{q} < 1$, recall (1.24). Combining this with Lemma 2.2.2, we conclude that $\widetilde{q} \in [-1, 1)$. Consequently, the coefficients in the recurrence (1.23) are uniformly bounded in n for any fixed t and x (since $[n]_{\widetilde{q}}$ is uniformly bounded in n for such $\widetilde{q}$). Therefore, the polynomials $\{\widetilde{W}_n(y; x, t)\}_{n=0}^{\infty}$ are orthogonal with respect to a boundedly supported measure (consult e.g. Theorems 2.5.4 and 2.5.5 in [36]). Hence, $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$ is determined by its moments.

⋇ $q = 1$, $\sigma = \tau = 0$. In this case $\xi_0 = 0$ (see (1.22)), which implies $\widetilde{b}_n = 0$ for all $n \in \mathbb{N}$. Consequently, the polynomials $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$ are orthogonal with respect to a Dirac measure (see Theorem A.1 in [22]), which is obviously determined by its moments.

As a result, in such cases, we can apply [12, Theorem 30.2], which states that under the moment determinacy, the convergence of moments implies weak convergence of probability measures. Therefore, we obtain the following corollary:

Corollary 3.2.1. *Probability measures* $\frac{(1+\sigma t)(y-x)^2}{h(1+\eta x+\sigma x^2)}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ *and* $\frac{(1+\sigma(t-h))(y-x)^2}{h(1+\eta x+\sigma x^2)}\mathbb{P}_{t-h,t}(x,\mathrm{d}y)$ *converge weakly to* $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$.

When $q=1-2\sqrt{\sigma\tau}$, and $\sigma\neq 0$ or $\tau\neq 0$, it is not known whether moment determinacy (and consequently weak convergence) holds. Therefore, in our further considerations, we will restrict ourselves to the parameters of quadratic harnesses that satisfy the conditions mentioned above.

3.3. Extension of the domain

In this section, we will show that not only polynomials belong to the domain of the infinitesimal generator of certain quadratic harnesses.

Theorem 3.3.1. *Let us consider* $QH(\eta,\theta;\sigma,\tau;q)$ *with* $q\in[-1,1-2\sqrt{\sigma\tau})$. *For any bounded continuous function* $g:\mathbb{R}\to\mathbb{R}$ *with a bounded continuous second derivative, we have*

$$\begin{aligned}(\mathbf{A}_t^{\pm}g)(x) = \tfrac{1+\eta x+\sigma x^2}{2(1+\sigma t)}&g''(x)\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\{x\})\\ &+\tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\int\limits_{\mathbb{R}\setminus\{x\}}\frac{\partial}{\partial x}\left(\frac{g(y)-g(x)}{y-x}\right)\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y)\end{aligned} \tag{3.2}$$

for x *satisfying* (3.1)*, where* $\nu_{x,t,\eta,\theta,\sigma,\tau,q}$ *is the probability measure defined in Theorem 1.6.1.*

Proof. We will prove (3.2) only for $\mathbf{A}_t^{+}$, because exactly the same arguments apply also to $\mathbf{A}_t^{-}$.
Fix x satisfying (3.1). Define a function $h_x:\mathbb{R}\to\mathbb{R}$ by

$$h_x(y):=\begin{cases}\frac{\partial}{\partial x}\frac{g(y)-g(x)}{y-x} & \text{for } y\neq x,\\ \frac{1}{2}g''(x) & \text{for } y=x.\end{cases}$$

Taylor's theorem yields

$$g(y)-g(x)=(y-x)g'(x)+\int_x^y g''(z)(y-z)\mathrm{d}z, \tag{3.3}$$

so for $y\neq x$ we get

$$h_x(y)=\tfrac{\partial}{\partial x}\left(g'(x)+\tfrac{1}{y-x}\int_x^y g''(z)(y-z)\mathrm{d}z\right)=\tfrac{1}{(y-x)^2}\int_x^y g''(z)(y-z)\mathrm{d}z. \tag{3.4}$$

Thus h_x is a bounded continuous function. Indeed, formula (3.4) gives

$$|h_x(y)| \leqslant \tfrac{1}{2}\sup_{y\in\mathbb{R}} |g''(y)|$$

and l'Hospital's rule implies that

$$\lim_{y\to x} h_x(y) = \tfrac{1}{2}g''(x).$$

Because of (3.3),

$$\int_{\mathbb{R}} \tfrac{g(y)-g(x)}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = g'(x)\int_{\mathbb{R}} \tfrac{y-x}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) + J(h,x), \tag{3.5}$$

where $J(h,x) := \int_{\mathbb{R}} \frac{\int_x^y g''(z)(y-z)\mathrm{d}z}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$. The integrand is zero when $y = x$, hence

$$J(h,x) = \int_{\mathbb{R}\backslash\{x\}} \tfrac{\int_x^y g''(z)(y-z)\mathrm{d}z}{(y-x)^2}\cdot\tfrac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \int_{\mathbb{R}\backslash\{x\}} h_x(y)\cdot\tfrac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y),$$

recall (3.4). Consequently, using again the fact that the integrand is zero when $y = x$, we get

$$J(h,x) = \int_{\mathbb{R}} h_x(y)\tfrac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y).$$

Therefore, the weak convergence of $\frac{(y-x)^2}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y)$ to $\frac{1+\eta x+\sigma x^2}{1+\sigma t}\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y)$, recall Corollary 3.2.1, gives

$$J(h,x) \xrightarrow{h\to 0^+} \tfrac{1+\eta x+\sigma x^2}{1+\sigma t}\int_{\mathbb{R}} h_x(y)\nu_{x,t,\eta,\theta,\sigma,\tau,q}(\mathrm{d}y).$$

Consequently, taking the limit $h \to 0^+$ in (3.5) yields (3.2) because Theorem 1.6.1 states that

$$\lim_{h\to 0^+}\int_{\mathbb{R}} \tfrac{y-x}{h}\mathbb{P}_{t,t+h}(x,\mathrm{d}y) = \mathbf{A}_t(x) = 0.$$

$\square$

It is worth mentioning that for all cases of quadratic harnesses $QH(\eta,\theta;\sigma,\tau;q)$ with $q \in [-1, 1-2\sqrt{\sigma\tau})$ such that for any fixed t and x, and all small enough h the supports of the transition probabilities $\mathbb{P}_{t,t+h}(x, \mathrm{d}y)$ and $\mathbb{P}_{t-h,t}(x, \mathrm{d}y)$ are contained in some interval, we obtain a stronger assertion of Theorem 3.3.1, namely that the domain of $\mathbf{A}_t^{\pm}$ contains $C^2(\mathbb{R})$. On the other hand, for $q \in [-1, 1-2\sqrt{\sigma\tau})$, all the cases of quadratic harnesses known from the literature satisfy the support conditions mentioned above (see [22] or Section 1.3 in the thesis).

From the proof of Theorem 3.3.1, we automatically obtain the same result for $q = 1$ and $\sigma = \tau = 0$, as the proof relies solely on the weak convergence which in turn is shown to hold in Section 3.2:

Theorem 3.3.2. *Consider $QH(\eta, \theta; 0, 0; 1)$. Then for any bounded continuous function $g : \mathbb{R} \to \mathbb{R}$ with a bounded continuous second derivative, formula* (3.2) *holds for all x satisfying* (3.1) *with $\sigma = \tau = 0$ and $q = 1$.*

In particular, Theorem 3.3.2 applies to the Wiener process (Example 1.2.2) and the standardized Poisson process (Example 1.2.3).

Chapter 4

Properties of the algebra $\mathcal{Q}$

In this chapter:

(i) certain subspaces of $\mathcal{Q}$ are defined in order to facilitate the analysis of the special elements of $\mathcal{Q}$ introduced in Chapter 2,

(ii) certain linear operators acting on $\mathcal{Q}$ are presented; they will be responsible for changing the order of operands in the multiplication (2.1).

Furthermore, we will focus on studying some properties of the introduced objects. The results presented in this chapter will play a crucial role in the process of removing Assumptions A1–A3 carried on in the subsequent two chapters.

4.1. Subspaces of the algebra $\mathcal{Q}$

In $\mathcal{Q}$, we consider a family of linear subspaces of $\mathcal{Q}$ given by:

$$\mathcal{Q}_k := \operatorname{span}\{\mathbb{F}^\ell \mathbb{D}^{\ell+k} : \ell \in \mathbb{N}_0\}, \qquad k \in \mathbb{N}_0.$$

From the definition (2.1) of multiplication, it is easy to see that $\mathcal{Q}_k$ contains only elements that have a monomial x^{n-k} with some coefficient (which can be zero) on their nth coordinate, $n \geqslant k$. Moreover, elements of $\mathcal{Q}_k$ have zeros on the first $k-1$ entries. In particular, $\mathbb{D}^k \in \mathcal{Q}_k$ (here and in the remainder of the thesis, we interpret $\mathbb{X}^k$ as $\mathbb{E}$ when $k = 0$ for any element $\mathbb{X} \in \mathcal{Q}$).

Next we define the left and right cosets of $\mathcal{Q}_0$:

$$\mathbb{X}\mathcal{Q}_0 := \{\mathbb{X}\mathbb{Y} : \mathbb{Y} \in \mathcal{Q}_0\} \qquad \text{and} \qquad \mathcal{Q}_0\mathbb{X} := \{\mathbb{Y}\mathbb{X} : \mathbb{Y} \in \mathcal{Q}_0\},$$

where $\mathbb{X} \in \mathcal{Q}$. We can express $\mathcal{Q}_k$ as certain left and right cosets of $\mathcal{Q}_0$.

Lemma 4.1.1. *For all $m \in \mathbb{N}_0$, we have*

$$\mathbb{D}^m \mathcal{Q}_0 = \mathcal{Q}_0 \mathbb{D}^m = \mathcal{Q}_m, \qquad \mathcal{Q}_m \mathbb{F}^m = \mathcal{Q}_0, \qquad \mathbb{F}^m \mathcal{Q}_m \subseteq \mathcal{Q}_0.$$

Proof. Let $\mathbb{X} \in \mathbb{D}^m \mathcal{Q}_0$. Then there exist coefficients $\{x_\ell\}_{\ell=0}^{\infty}$ such that $\mathbb{X} = \sum_{\ell=0}^{\infty} x_\ell \mathbb{D}^m \mathbb{F}^\ell \mathbb{D}^\ell$. Then (2.4) implies

$$\mathbb{X} = \sum_{\ell=0}^{m} x_\ell \mathbb{D}^m + \sum_{\ell=m+1}^{\infty} x_\ell \mathbb{F}^{\ell-m} \mathbb{D}^\ell = \sum_{\ell=0}^{m} x_\ell \mathbb{D}^m + \sum_{\ell=1}^{\infty} x_{\ell+m} \mathbb{F}^\ell \mathbb{D}^{\ell+m}.$$

This representation shows that $\mathbb{X} \in \mathcal{Q}_m$. Conversely, for $\mathbb{X} \in \mathcal{Q}_m$, we can find coefficients $\{\widetilde{x}_\ell\}_{\ell=0}^{\infty}$ such that

$$\mathbb{X} = \sum_{\ell=0}^{\infty} \widetilde{x}_\ell \mathbb{F}^\ell \mathbb{D}^{\ell+m} = \mathbb{D}^m \sum_{\ell=0}^{\infty} \widetilde{x}_\ell \mathbb{F}^{\ell+m} \mathbb{D}^{\ell+m} = \mathbb{D}^m \sum_{\ell=m}^{\infty} \widetilde{x}_{\ell-m} \mathbb{F}^\ell \mathbb{D}^\ell.$$

Hence, we can conclude that $\mathbb{X} \in \mathbb{D}^m \mathcal{Q}_0$. The remaining cases can be proved in a similar way. $\square$

Clearly, $\mathbb{F}^0 \mathcal{Q}_0 = \mathcal{Q}_0$. On the other hand, there exist no $m \in \mathbb{N}$ such that $\mathbb{F}^m \mathcal{Q}_m = \mathcal{Q}_0$. To see this, consider $\mathbb{E} \in \mathcal{Q}_0$. If $\mathbb{F}^m \mathbb{X} = \mathbb{E}$ for some $\mathbb{X} \in \mathcal{Q}_m$, then using (2.4), we would have $\mathbb{X} = \mathbb{D}^m$. However, this leads to a contradiction since

$$\mathbb{F}^m \mathbb{X} = \mathbb{F}^m \mathbb{D}^m = (\underbrace{0, \ldots, 0}_{m \text{ times}}, x^m, x^{m+1}, x^{m+2}, \ldots),$$

which is not equal to $\mathbb{E}$.

Next, we can determine the subspace to which a product of elements from the given subspaces belongs. Namely:

Remark 4.1.2. *If $\mathbb{X} \in \mathcal{Q}_k$ and $\mathbb{Y} \in \mathcal{Q}_\ell$, where $k, \ell \in \mathbb{N}_0$, then $\mathbb{X}\mathbb{Y} \in \mathcal{Q}_{k+\ell}$. Furthermore, $\mathbb{X}\mathbb{Y} = \mathbb{Y}\mathbb{X}$ when $k = \ell = 0$.*

Proof. First, we will prove the case $k = \ell = 0$. For $0 \leqslant m \leqslant n$, using (2.4), we have

$$\mathbb{F}^m \mathbb{D}^m \mathbb{F}^n \mathbb{D}^n = \mathbb{F}^{m+n-m} \mathbb{D}^n = \mathbb{F}^n \mathbb{D}^n = \mathbb{F}^n \mathbb{D}^{n-m+m} = \mathbb{F}^n \mathbb{D}^n \mathbb{F}^m \mathbb{D}^m, \tag{4.1}$$

which shows that $\mathbb{F}^m\mathbb{D}^m$ and $\mathbb{F}^n\mathbb{D}^n$ commute.

Now, let $\mathbb{X}$ and $\mathbb{Y}$ be arbitrary elements from $\mathcal{Q}_0$, which can be written as $\mathbb{X} = \sum_{n=0}^{\infty} x_n\mathbb{F}^n\mathbb{D}^n \in \mathcal{Q}_0$ and $\mathbb{Y} = \sum_{m=0}^{\infty} y_m\mathbb{F}^m\mathbb{D}^m \in \mathcal{Q}_0$ with coefficients $\{x_n\}_{n=0}^{\infty}$ and $\{y_m\}_{m=0}^{\infty}$, respectively. Then, according to (4.1),

$$\mathbb{X}\mathbb{Y} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_n y_m \mathbb{F}^n\mathbb{D}^n\mathbb{F}^m\mathbb{D}^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_n y_m \mathbb{F}^m\mathbb{D}^m\mathbb{F}^n\mathbb{D}^n = \mathbb{Y}\mathbb{X}.$$

Additionally, the second equality in (4.1) shows that $\mathbb{X}\mathbb{Y} \in \mathcal{Q}_0$. In fact,

$$\mathbb{X}\mathbb{Y} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} x_n y_m \mathbb{F}^n\mathbb{D}^n + \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} x_n y_m \mathbb{F}^m\mathbb{D}^m \in \mathcal{Q}_0.$$

If either k or ℓ is nonzero, then from Lemma 4.1.1, we know that there exist $\widetilde{\mathbb{X}} \in \mathcal{Q}_0$ and $\widetilde{\mathbb{Y}} \in \mathcal{Q}_0$ such that $\mathbb{X} = \mathbb{D}^k\widetilde{\mathbb{X}}$ and $\mathbb{Y} = \mathbb{D}^\ell\widetilde{\mathbb{Y}}$. Moreover, the first equality in Lemma 4.1.1 implies that $\widetilde{\mathbb{X}}\mathbb{D}^\ell = \mathbb{D}^\ell\breve{\mathbb{X}}$ for some $\breve{\mathbb{X}} \in \mathcal{Q}_0$. Consequently,

$$\mathbb{X}\mathbb{Y} = \mathbb{D}^k\widetilde{\mathbb{X}}\mathbb{D}^\ell\widetilde{\mathbb{Y}} = \mathbb{D}^{k+\ell}(\breve{\mathbb{X}}\widetilde{\mathbb{Y}}),$$

where the term in the brackets belongs to $\mathcal{Q}_0$ according to the first part of our arguments in this proof. Consequently, $\mathbb{X}\mathbb{Y} \in \mathbb{D}^{k+\ell}\mathcal{Q}_0$, and Lemma 4.1.1 ends the proof. □

Note that the second part of the above remark implies that $\mathcal{Q}_0$ is a commutative subalgebra of $\mathcal{Q}$. This fact will be extensively used throughout the remainder of the thesis. In addition, Lemma 4.1.1 and Remark 4.1.2 will be used to quickly determine the subspace to which a given element belongs. These results will be extensively used further without direct reference.

Moreover, let us state the following lemma for future reference:

Lemma 4.1.3. *For each $0 \leqslant \ell \leqslant k$ and $\mathbb{X} \in \mathcal{Q}_k$, we have*

$$\mathbb{X}\mathbb{F}^\ell\mathbb{D}^\ell = \mathbb{X}.$$

Proof. Note that for $0 \leqslant \ell \leqslant k$, identity (2.4) implies that

$$\mathbb{X}\mathbb{F}^\ell\mathbb{D}^\ell = \sum_{m=0}^{\infty} x_m\mathbb{F}^m\mathbb{D}^{m+k}\mathbb{F}^\ell\mathbb{D}^\ell = \sum_{m=0}^{\infty} x_m\mathbb{F}^m\mathbb{D}^{m+k} = \mathbb{X},$$

where we have represented $\mathbb{X}$ as $\sum_{m=0}^{\infty} x_m \mathbb{F}^m \mathbb{D}^{m+k}$ with some coefficients $\{x_m\}_{m=0}^{\infty}$. $\square$

A linear combination of the identities presented in Lemma 4.1.3 leads to the following conclusion:

Corollary 4.1.4. *Consider some real numbers $a_0, \ldots, a_k$ satisfying $\sum_{\ell=0}^{k} a_\ell = 0$, $k \in \mathbb{N}_0$. Then for $\mathbb{X} \in \mathcal{Q}_k$ and $\mathbb{Y} = \sum_{\ell=0}^{k} a_\ell \mathbb{F}^\ell \mathbb{D}^\ell$*

$$\mathbb{X}\mathbb{Y} = \mathbb{0}.$$

We will close this section with two remarks on the invertibility of elements of $\mathcal{Q}$. Firstly, according to [24, Proposition 1.2], it is clear that all elements of $\mathcal{Q}_0$ with all nonzero coordinates are invertible. Secondly, let us single out the following observation for further reference:

Remark 4.1.5. *Assume that $\mathbb{X}_j \in \mathcal{Q}_j$ for all $j \in \mathbb{N}_0$ and that $\mathbb{X}_0$ is invertible. Then $\mathbb{X} := \sum_{j=0}^{\infty} \mathbb{X}_j$ is also invertible.*

The above fact follows also from Proposition 1.2 in [24].

4.2. Linear operators on $\mathcal{Q}$

We will now introduce certain linear operators that facilitate changing the order of operands in the multiplication (2.1).

4.2.1. Linear operator $\mathcal{S}$

Define a linear operator $\mathcal{S} : \mathcal{Q} \to \mathcal{Q}$ as follows:

$$\mathcal{S}(\mathbb{X}) := \mathbb{D}\mathbb{X}\mathbb{F}. \tag{4.2}$$

Lemma 4.1.1 implies that $\mathcal{S} : \mathcal{Q}_k \to \mathcal{Q}_k$. In the following, we will consider the k-fold composition of $\mathcal{S}$ denoted as $\mathcal{S}^k$, $k \in \mathbb{N}_0$. For $k = 0$, we interpret it as the identity operator.

It turns out that $\mathcal{S}$ acts on a product of elements from the linear subspaces introduced in Section 4.1 in the following way:

Remark 4.2.1. *For $\mathbb{X} \in \mathcal{Q}_k$ and $\mathbb{Y} \in \mathcal{Q}_\ell$, where $k, \ell \in \mathbb{N}_0$,*

$$\mathcal{S}(\mathbb{X}\mathbb{Y}) = \mathcal{S}(\mathbb{X})\mathcal{S}(\mathbb{Y}).$$

Proof. Note that $\mathbb{D}\mathbb{X} \in \mathcal{Q}_{k+1}$. Lemma 4.1.3 gives that $\mathbb{D}\mathbb{X} = \mathbb{D}\mathbb{X}\mathbb{F}\mathbb{D}$. Consequently,

$$\mathcal{S}(\mathbb{X}\mathbb{Y}) = \mathbb{D}\mathbb{X}\mathbb{Y}\mathbb{F} = \mathbb{D}\mathbb{X}\mathbb{F}\mathbb{D}\mathbb{Y}\mathbb{F} = \mathcal{S}(\mathbb{X})\mathcal{S}(\mathbb{Y}).$$

□

Hence, for any invertible $\mathbb{X} \in \mathcal{Q}$, we have

$$\mathcal{S}(\mathbb{X})\mathcal{S}(\mathbb{X}^{-1}) = \mathcal{S}(\mathbb{X}^{-1})\mathcal{S}(\mathbb{X}) = \mathbb{E}. \tag{4.3}$$

Therefore, there exists the inverse of $\mathcal{S}(\mathbb{X})$ satisfying

$$(\mathcal{S}(\mathbb{X}))^{-1} = \mathcal{S}(\mathbb{X}^{-1}). \tag{4.4}$$

It turns out that the operator $\mathcal{S}$ is useful in a task of changing the order of multiplication of some elements of $\mathcal{Q}$:

Lemma 4.2.2. *Let $\mathbb{X} \in \mathcal{Q}_0$ and $\mathbb{Y} \in \mathcal{Q}_k$, $k \in \mathbb{N}_0$. Then*

$$\mathcal{S}^k(\mathbb{X})\mathbb{Y} = \mathbb{Y}\mathbb{X}.$$

Proof. Lemma 4.1.1 yields that $\mathbb{F}^k\mathbb{Y} \in \mathcal{Q}_0$. The second part of Remark 4.1.2 used with $\mathbb{X} \in \mathcal{Q}_0$ and $\mathbb{F}^k\mathbb{Y} \in \mathcal{Q}_0$ implies that $\mathbb{X}$ and $\mathbb{F}^k\mathbb{Y}$ commute. As a result,

$$\mathcal{S}^k(\mathbb{X})\mathbb{Y} = \mathbb{D}^k\mathbb{X}\mathbb{F}^k\mathbb{Y} = \mathbb{D}^k(\mathbb{F}^k\mathbb{Y})\mathbb{X} = \mathbb{Y}\mathbb{X}.$$

□

Using the properties of $\mathcal{S}$ we can also deduce the following:

Lemma 4.2.3. *Let $\mathbb{X}, \mathbb{Y} \in \mathcal{Q}_k$, $k \in \mathbb{N}_0$. Then*

$$\mathbb{X}\mathbb{F}^k\mathbb{Y} = \mathbb{Y}\mathbb{F}^k\mathbb{X}.$$

Proof. Lemma 4.2.2 used with $\mathbb{Y} \in \mathcal{Q}_k$ and $\mathbb{F}^k\mathbb{X} \in \mathcal{Q}_0$ gives

$$\mathbb{Y}\mathbb{F}^k\mathbb{X} = \mathcal{S}^k(\mathbb{F}^k\mathbb{X})\mathbb{Y} = \mathbb{X}\mathbb{F}^k\mathbb{Y}.$$

The last equality holds due to the definition of $\mathcal{S}$ and the relation (2.4) between $\mathbb{D}$ and $\mathbb{F}$. □

Moreover, the operators $\mathcal{S}$ and $\mathcal{R}$ sometimes commute.

Lemma 4.2.4. *For every $k \in \mathbb{N}$, the operators $\mathcal{S}$ and $\mathcal{R}$ commute on $\mathcal{Q}_k$, where $\mathcal{R}$ is defined in* (2.24)*.*

Proof. Let $\mathbb{X} \in \mathcal{Q}_k$ be arbitrary. Using the definition of $\mathcal{R}$, we get

$$\mathcal{R}(\mathcal{S}(\mathbb{X})) = \mathbb{E} + \eta\mathcal{S}(\mathbb{X})\mathbb{F} + \sigma(\mathcal{S}(\mathbb{X})\mathbb{F})^2 = \mathbb{E} + \eta\mathbb{D}(\mathbb{X}\mathbb{F})\mathbb{F} + \sigma\mathcal{S}(\mathbb{X})\mathbb{F}\mathbb{D}\mathbb{X}\mathbb{F}^2.$$

Since $\mathcal{S}(\mathbb{X}) \in \mathcal{Q}_k$, Lemma 4.1.3 says that $\mathcal{S}(\mathbb{X})\mathbb{F}\mathbb{D} = \mathcal{S}(\mathbb{X})$. Consequently,

$$\mathcal{R}(\mathcal{S}(\mathbb{X})) = \mathbb{E} + \eta\mathcal{S}(\mathbb{X}\mathbb{F}) + \sigma\mathcal{S}(\mathbb{X})\mathbb{X}\mathbb{F}^2 = \mathbb{E} + \eta\mathcal{S}(\mathbb{X}\mathbb{F}) + \sigma\mathbb{D}(\mathbb{X}\mathbb{F})^2\mathbb{F} = \mathcal{S}(\mathcal{R}(\mathbb{X})).$$

□

4.2.2. Linear operator $\mathcal{T}$

Now we define a linear operator $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ by the following formula:

$$\mathcal{T}(\mathbb{X}) := \mathbb{F}\mathbb{X}\mathbb{D}.$$

Lemma 4.1.1 yields that $\mathcal{T} : \mathcal{Q}_k \to \mathcal{Q}_k$. Moreover, directly from the definitions of operators $\mathcal{S}$ and $\mathcal{T}$ and identity (2.4) we obtain $\mathcal{S}(\mathcal{T}(\mathbb{X})) = \mathbb{X}$.
In general, $\mathcal{T}(\mathcal{S}(\mathbb{X})) \neq \mathbb{X}$. However, we can prove the following:

Lemma 4.2.5. *If $\mathbb{X} \in \mathcal{Q}_k$ and $\mathbb{Y} \in \mathcal{Q}_\ell$, $k, \ell \in \mathbb{N}$, then*

$$\mathbb{X}\mathcal{T}(\mathcal{S}(\mathbb{Y})) = \mathbb{X}\mathbb{Y}.$$

Proof. Since $\mathbb{X} \in \mathcal{Q}_k$, $k \in \mathbb{N}$, we get that $\mathbb{X}\mathbb{F}\mathbb{D} = \mathbb{X}$ by Lemma 4.1.3. Analogously, $\mathbb{Y}\mathbb{F}\mathbb{D} = \mathbb{Y}$. As a result,

$$\mathbb{X}\mathcal{T}(\mathcal{S}(\mathbb{Y})) = \mathbb{X}\mathbb{F}(\mathbb{D}\mathbb{Y}\mathbb{F})\mathbb{D} = (\mathbb{X}\mathbb{F}\mathbb{D})(\mathbb{Y}\mathbb{F}\mathbb{D}) = \mathbb{X}\mathbb{Y}.$$

□

Moreover, with help of $\mathcal{S}$ and $\mathcal{T}$ we can easily change the order of multiplication of elements of the subspaces $\mathcal{Q}_k$:

Lemma 4.2.6. *For $\mathbb{X} \in \mathcal{Q}_k$ and $\mathbb{Y} \in \mathcal{Q}_\ell$, $k, \ell \in \mathbb{N}_0$, we have*

$$\mathbb{X}\mathbb{Y} = \mathcal{S}^k(\mathbb{Y})\mathcal{T}^\ell(\mathbb{X}).$$

Proof. Lemma 4.2.2 applied to $\mathbb{Y} \in \mathcal{Q}_\ell$ and $\mathbb{F}^{k+\ell}\mathbb{X}\mathbb{D}^\ell \in \mathcal{Q}_0$ gives

$$\mathcal{S}^k(\mathbb{Y})\mathcal{T}^\ell(\mathbb{X}) = \mathbb{D}^k\mathbb{Y}\mathbb{F}^{k+\ell}\mathbb{X}\mathbb{D}^\ell = \mathbb{D}^k\mathcal{S}^\ell(\mathbb{F}^{k+\ell}\mathbb{X}\mathbb{D}^\ell)\mathbb{Y} = \mathbb{X}\mathbb{Y}$$

(above we have used (2.4)). □

4.2.3. Commutator

Apart from $\mathcal{S}$ and $\mathcal{T}$, we will also use the commutator:

$$[\mathbb{X}, \mathbb{Y}] := \mathbb{X}\mathbb{Y} - \mathbb{Y}\mathbb{X}, \qquad \mathbb{X}, \mathbb{Y} \in \mathcal{Q}.$$

It is obvious that for all $\mathbb{X}, \mathbb{Y}, \mathbb{Y}_1, \mathbb{Y}_2 \in \mathcal{Q}$ and $\alpha, \beta \in \mathbb{R}$

$$[\mathbb{X}, \alpha\mathbb{Y}_1 + \beta\mathbb{Y}_2] = \alpha[\mathbb{X}, \mathbb{Y}_1] + \beta[\mathbb{X}, \mathbb{Y}_2], \tag{4.5}$$

and

$$[\mathbb{X}, \mathbb{Y}] = -[\mathbb{Y}, \mathbb{X}].$$

Moreover, in view of Remark 4.2.1,

$$\mathcal{S}([\mathbb{X}, \mathbb{Y}]) = [\mathcal{S}(\mathbb{X}), \mathcal{S}(\mathbb{Y})]. \tag{4.6}$$

Furthermore, using the definition of the commutator along with Remark 4.1.2, we have:

Remark 4.2.7. *For* $\mathbb{X}, \mathbb{Y} \in \mathcal{Q}_0$

$$[\mathbb{X}, \mathbb{Y}] = \mathbb{O}.$$

In particular,

$$[\mathbb{X}, \mathbb{E}] = \mathbb{O}. \tag{4.7}$$

Moreover, the commutator has the following properties:

Lemma 4.2.8. *For all* $\mathbb{X}, \mathbb{Y} \in \mathcal{Q}$

$$[\mathbb{X}\mathbb{Y}, \mathbb{X}] = \mathbb{X}[\mathbb{Y}, \mathbb{X}] \qquad \textit{and} \qquad [\mathbb{X}\mathbb{Y}, \mathbb{Y}] = [\mathbb{X}, \mathbb{Y}]\mathbb{Y}.$$

Proof. The definition of the commutator implies that

$$[\mathbb{X}\mathbb{Y}, \mathbb{X}] = \mathbb{X}\mathbb{Y}\mathbb{X} - \mathbb{X}^2\mathbb{Y} = \mathbb{X}(\mathbb{Y}\mathbb{X} - \mathbb{X}\mathbb{Y}) = \mathbb{X}[\mathbb{Y}, \mathbb{X}].$$

We proceed similarly with the proof of the second assertion. □

In addition, we present below an observation that will be used later several times:

Remark 4.2.9. *For any* $\mathbb{X}, \mathbb{Y} \in \mathcal{Q}$ *and* $\alpha \in \mathbb{R}$ *we have*

$$(\mathbb{E} + \alpha\mathbb{Y})\mathbb{X}\mathbb{Y} - \mathbb{Y}\mathbb{X}(\mathbb{E} + \alpha\mathbb{Y}) = [\mathbb{X}, \mathbb{Y}].$$

Chapter 5

Some important elements of $\mathcal{Q}$

In this chapter, we are going to introduce some special elements of $\mathcal{Q}$ that will prove to be very useful in our task of removing Assumptions A1–A3.
Recall that we are working under assumptions (1.9) with $\sigma\tau = 0$. Define

$$\kappa_1 := \sigma t(1 - q + \sigma t). \tag{5.1}$$

Using (2.19) and the fact that $\sigma\tau = 0$, we have

$$\kappa_1 = \sigma\kappa_0. \tag{5.2}$$

Chapter 5 is divided into two sections.

Section 5.1 introduces a number of special elements of $\mathcal{Q}$ which will serve as building blocks for some more complicated elements. Even though it might be hard to grasp the importance of these elements within this chapter, their usefulness will become clear in Chapter 6, when we will be able to represent $\mathbb{S}(z,t)$, recall (2.26) and (2.27), in their terms.

The aim of Section 5.2 is to provide some explicit formulas for the elements $\mathbb{U}(z,t)$ and $\mathbb{Y}(z,t)$, which appear in Assumption A1. Again, these formulas will be based on the building blocks presented in Section 5.1.

Undoubtedly, the content of this chapter is quite complicated, so in order to make it easier to digest, we provide a diagram in Figure 5.1. The diagram presents the main results of the next two chapters along with their interdependencies.

Recall that our primary objective is to prove the identities from Assumptions A1–A3 (highlighted in gray in the diagram).

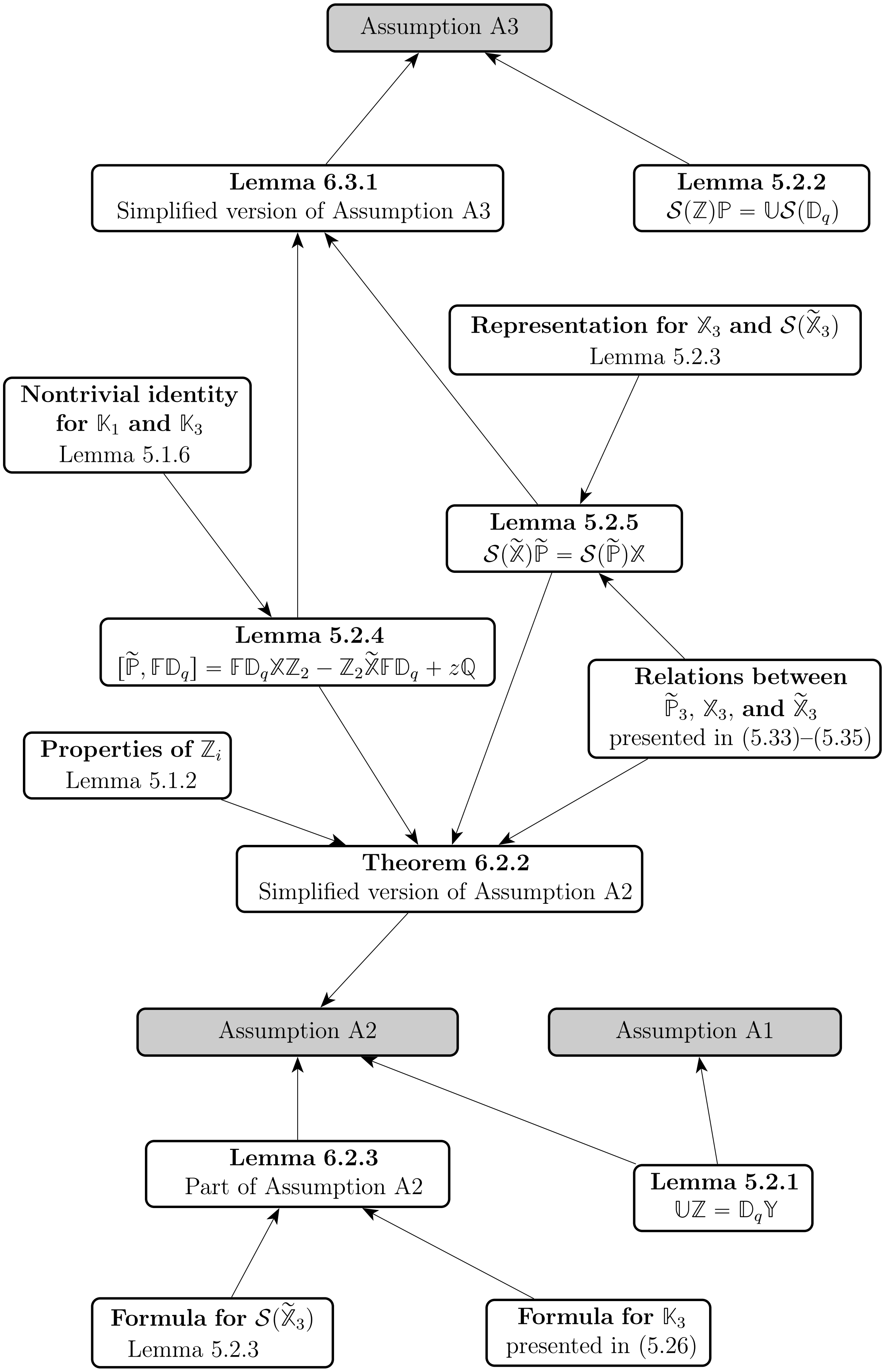

Figure 5.1: Scheme of relations between the main lemmas and theorems

Throughout Chapter 5, $z \in \mathbb{R}$ and $t \geqslant 0$ are some fixed, but arbitrary, parameters. For simplicity, we will suppress them as arguments, so, for example, we will write $\mathbb{U}$ instead of $\mathbb{U}(z,t)$, even though $\mathbb{U}$ in fact depends on z and t.
In addition, we will make extensive use of the tools presented in Chapter 4. Occasionally, for brevity, we will use these tools without explicit reference (especially Remark 4.2.1 and Remark 4.1.2).

5.1. Basic elements of $\mathcal{Q}$

5.1.1. Element $\mathbb{D}_q$

The definition of $\mathbb{D}_q$ was given in (2.6), which yields that $\mathbb{D}_q \in \mathcal{Q}_1$; $\mathbb{D}_q$ can also be represented coordinate-wise as in (2.7).

Lemma 5.1.1. *Element $\mathbb{D}_q$ satisfies the following equation:*

$$\mathbb{D}_q\mathbb{F} = q\mathbb{F}\mathbb{D}_q + \mathbb{E}. \tag{5.3}$$

Proof. Using (2.4) and (2.6) we get

$$\mathbb{D}_q\mathbb{F} - q\mathbb{F}\mathbb{D}_q = \sum_{k=0}^{\infty} q^k\mathbb{F}^k\mathbb{D}^k - \sum_{k=1}^{\infty} q^k\mathbb{F}^k\mathbb{D}^k = \mathbb{E}.$$

□

Formula (5.3) can be rewritten as

$$[\mathbb{D}_q, \mathbb{F}] = (q-1)\mathbb{F}\mathbb{D}_q + \mathbb{E}. \tag{5.4}$$

5.1.2. Elements $\mathbb{Z}_i$, $i = 0, 1, 2, 3$

Let us consider the elements

$$\mathbb{Z}_0 := \mathbb{E} + \sigma t\mathbb{F}^3\mathbb{D}_q\mathbb{D}^2, \qquad \mathbb{Z}_k := \mathcal{S}^k(\mathbb{Z}_0), \qquad k = 1, 2, 3,$$

recall the definition (4.2) of $\mathcal{S}$. All $\mathbb{Z}_i$, $i = 0, 1, 2, 3$, belong to $\mathcal{Q}_0$. It is easy to see that

$$\mathbb{Z}_k = \mathbb{E} + \sigma t \mathbb{F}^{3-k}\mathbb{D}_q\mathbb{D}^{3-k}\mathbb{F}, \qquad k = 0, 1, 2, 3. \tag{5.5}$$

Hence, we notice that the nth coordinate of $\mathbb{Z}_k$ is equal to $(1+\sigma t[n+k-2]_q)x^n$, recall (2.7) and the convention that $[n]_q = 0$ for $n \leqslant 0$. Under the assumptions on the parameters, we obtain that $\sigma t \geqslant 0$ and $[n]_q \geqslant 0$ for all $n \in \mathbb{N}_0$ (since $q \in [-1, 1]$). As a result, the coefficient $1 + \sigma t[n + k - 2]_q$ is nonzero for all $n \in \mathbb{N}_0$, so $\mathbb{Z}_k$ is invertible, $k = 0, 1, 2, 3$, which is assured by [24, Proposition 1.2].
Below we present some important identities that are satisfied by $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Lemma 5.1.2 (Properties of $\mathbb{Z}_i$, i=2,3)**.** *The following identities hold:*

$$\mathbb{Z}_2 + (q - \sigma t)\mathbb{F}\mathbb{D}_q = \mathbb{D}_q\mathbb{F}, \tag{5.6}$$

$$(1 + \sigma t)\mathbb{Z}_2 = \mathbb{Z}_3 + \kappa_1\mathbb{F}\mathbb{D}_q, \tag{5.7}$$

$$\mathbb{Z}_3 = (1 + \sigma t)\mathbb{D}_q\mathbb{F} - q\mathbb{F}\mathbb{D}_q, \tag{5.8}$$

$$(q - \sigma t)\mathbb{Z}_3 + \kappa_1\mathbb{D}_q\mathbb{F} = q\mathbb{Z}_2. \tag{5.9}$$

Proof. Formulas (5.5) (for $k = 2$) and (5.3) give (5.6) since

$$\mathbb{Z}_2 + (q - \sigma t)\mathbb{F}\mathbb{D}_q = \mathbb{E} + q\mathbb{F}\mathbb{D}_q = \mathbb{D}_q\mathbb{F}.$$

Assertion (5.7) follows from (5.1) together with (5.3) and (5.5) for $k = 3$:

$$(1 + \sigma t)\mathbb{Z}_2 - \kappa_1\mathbb{F}\mathbb{D}_q = (1 + \sigma t)\mathbb{E} + q\sigma t\mathbb{F}\mathbb{D}_q = \mathbb{E} + \sigma t(\mathbb{E} + q\mathbb{F}\mathbb{D}_q) = \mathbb{E} + \sigma t\mathbb{D}_q\mathbb{F} = \mathbb{Z}_3.$$

Furthermore, applying the same identities as above, we obtain (5.8) and (5.9) as

$$(1 + \sigma t)\mathbb{D}_q\mathbb{F} - q\mathbb{F}\mathbb{D}_q = \mathbb{D}_q\mathbb{F} - q\mathbb{F}\mathbb{D}_q + \sigma t\mathbb{D}_q\mathbb{F} = \mathbb{Z}_3$$

and

$$(q - \sigma t)\mathbb{Z}_3 + \kappa_1\mathbb{D}_q\mathbb{F} = (q - \sigma t)\mathbb{E} + \sigma t\mathbb{D}_q\mathbb{F} = q\mathbb{E} + \sigma t(\mathbb{D}_q\mathbb{F} - \mathbb{E}) = q\mathbb{Z}_2$$

– in the last step we have used (5.3) and (5.5) for $k = 2$. $\square$

Observe that the element $\mathbb{Z}$ given by (2.30) can be expressed in terms of $\mathbb{Z}_2$ as

$$\mathbb{Z} = \mathbb{Z}_2\mathbb{D}_q.$$

Thus $\mathbb{Z} \in \mathcal{Q}_1$ by Remark 4.1.2.

Furthermore, since $\mathbb{Z}_2 = \mathcal{S}(\mathbb{Z}_1)$, Lemma 4.2.2 implies

$$\mathbb{Z} = \mathbb{Z}_2\mathbb{D}_q = \mathcal{S}(\mathbb{Z}_1)\mathbb{D}_q = \mathbb{D}_q\mathbb{Z}_1. \tag{5.10}$$

5.1.3. Element $\mathbb{Q}$

We will also consider an element $\mathbb{Q} \in \mathcal{Q}$ given by

$$\mathbb{Q} := \mathbb{D}_q - \mathbb{F}\mathbb{D}_q\mathbb{D} \in \mathcal{Q}_1.$$

It turns out that the operator $\mathcal{S}$ acts nicely on $\mathbb{Q}$:

Lemma 5.1.3. *The following identity is satisfied:*

$$\mathcal{S}(\mathbb{Q}) = q\mathbb{Q}. \tag{5.11}$$

Proof. In view of (2.9) and (5.3),

$$\mathbb{D}_q = q\mathbb{F}\mathbb{D}_q\mathbb{D} + \mathbb{D} \qquad \text{and} \qquad \mathbb{Q} = [\mathbb{D}_q, \mathbb{F}]\mathbb{D}.$$

Consequently, from (5.4),

$$\mathcal{S}(\mathbb{Q}) = \mathbb{D}[\mathbb{D}_q, \mathbb{F}] = (q-1)\mathbb{D}_q + \mathbb{D} = q\mathbb{Q} + q\mathbb{F}\mathbb{D}_q\mathbb{D} + \mathbb{D} - \mathbb{D}_q = q\mathbb{Q}.$$

$\square$

It is worth noting that $\mathbb{Q}$ naturally arises as a result of taking the commutator of $\mathbb{F}\mathbb{D}_q$ and any element $\mathbb{X}$ of the subspace $\mathcal{Q}_1$. Indeed, Lemma 4.2.2 applied to $\mathbb{X} \in \mathcal{Q}_1$ and $\mathbb{F}^2\mathbb{D}_q\mathbb{D} \in \mathcal{Q}_0$ yields

$$[\mathbb{X}, \mathbb{F}\mathbb{D}_q] = \mathbb{X}\mathbb{F}\mathbb{D}_q - \mathbb{F}\mathbb{D}_q\mathbb{X} = \mathbb{X}\mathbb{F}(\mathbb{D}_q - \mathbb{F}\mathbb{D}_q\mathbb{D}) = \mathbb{X}\mathbb{F}\mathbb{Q}. \tag{5.12}$$

Furthermore, we have another representation of $\mathbb{Q}$ in terms of $\mathbb{Z}$ and $\mathbb{Z}_2$:

$$\mathbb{Q} = \mathbb{Z} - \mathbb{F}\mathbb{D}_q\mathbb{D}\mathbb{Z}_2. \tag{5.13}$$

This follows from Remark 4.2.9 applied to $\mathbb{D}$, $\mathbb{F}\mathbb{D}_q$, along with $\mathbb{Z}_2$:

$$\mathbb{Q} = [\mathbb{D}, \mathbb{F}\mathbb{D}_q] = \mathbb{Z}_2\mathbb{D}_q - \mathbb{F}\mathbb{D}_q\mathbb{D}\mathbb{Z}_2 = \mathbb{Z} - \mathbb{F}\mathbb{D}_q\mathbb{D}\mathbb{Z}_2.$$

5.1.4. Elements $\mathbb{T}_i$, $i = 1, 2, 3$

Let us introduce three other elements of $\mathcal{Q}$:

$$\mathbb{T}_1 := \mathbb{Z}_1^2 - \kappa_1(\mathbb{F}^2\mathbb{D}_q\mathbb{D})^2, \tag{5.14}$$

$$\mathbb{T}_2 := \mathcal{S}(\mathbb{T}_1) = \mathbb{Z}_2^2 - \kappa_1(\mathbb{F}\mathbb{D}_q)^2, \tag{5.15}$$

$$\mathbb{T}_3 := \mathbb{Z}_1\mathbb{Z}_2 - \kappa_1\mathbb{F}^2\mathbb{D}_q^2. \tag{5.16}$$

Note that they all belong to $\mathcal{Q}_0$, and the nth coordinate of each of them is a monomial of degree n with a nonzero leading coefficient, which (after simplification) is equal to

$$1 + \sigma t[2n-2]_q, \qquad 1 + \sigma t[2n]_q, \qquad \text{and} \qquad 1 + \sigma t[2n-1]_q, \tag{5.17}$$

respectively. In deriving these coefficients, we have used (5.1) and (1.20). Consequently, Proposition 1.2 in [24] implies that $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ are invertible elements of $\mathcal{Q}$, since $\sigma t \geqslant 0$ and $q \in [-1, 1]$ (which implies that $[n]_q \geqslant 0$ for all $n \geqslant 0$).
Now we will present some properties of these elements.

Lemma 5.1.4. *The following representation of $\mathbb{T}_2$ holds true:*

$$\mathbb{T}_2 = \mathbb{D}_q\mathbb{F}\mathbb{Z}_2 - q\mathbb{F}\mathbb{Z}. \tag{5.18}$$

Proof. Formula (5.6) implies:

$$\begin{aligned}\mathbb{D}_q\mathbb{F}\mathbb{Z}_2 - \mathbb{T}_2 &= (\mathbb{D}_q\mathbb{F} - \mathbb{Z}_2)\mathbb{Z}_2 + \kappa_1(\mathbb{F}\mathbb{D}_q)^2 = (q - \sigma t)\mathbb{F}\mathbb{D}_q\mathbb{Z}_2 + \kappa_1(\mathbb{F}\mathbb{D}_q)^2 \\ &= \mathbb{F}\mathbb{D}_q[(q - \sigma t)\mathbb{Z}_2 + \kappa_1\mathbb{F}\mathbb{D}_q].\end{aligned}$$

Lemma 4.2.2 applied to $\mathbb{D}_q \in \mathcal{Q}_1$ and to the expression in the square brackets (which represents an element of $\mathcal{Q}_0$) yields

$$\mathbb{D}_q\mathbb{F}\mathbb{Z}_2 - \mathbb{T}_2 = \mathbb{F}\mathcal{S}((q-\sigma t)\mathbb{Z}_2 + \kappa_1\mathbb{F}\mathbb{D}_q)\mathbb{D}_q = \mathbb{F}((q-\sigma t)\mathbb{Z}_3 + \kappa_1\mathbb{D}_q\mathbb{F})\mathbb{D}_q = q\mathbb{F}\mathbb{Z}_2\mathbb{D}_q,$$

where the last equality follows from (5.9). □

Lemma 5.1.5. *Element $\mathbb{T}_3$ satisfies the following equations:*

$$\mathcal{S}(\mathbb{T}_3) = \mathcal{S}(\mathbb{F}\mathbb{D}_q\mathbb{Z}_2 - q\mathbb{F}^2\mathbb{D}_q\mathbb{D}\mathbb{Z}_1), \tag{5.19}$$

$$\mathcal{S}^2(\mathbb{T}_3) = \mathcal{S}^2(\mathbb{D}_q\mathbb{F}\mathbb{Z}_1 - q\mathbb{F}\mathbb{D}_q\mathbb{Z}_0). \tag{5.20}$$

Proof. By the definition of $\mathbb{T}_3$, we get

$$\mathbb{T}_3 - \mathbb{F}\mathbb{D}_q\mathbb{Z}_2 + q\mathbb{F}^2\mathbb{D}_q\mathbb{D}\mathbb{Z}_1 = (\mathbb{Z}_1 - \mathbb{F}\mathbb{D}_q)\mathbb{Z}_2 - \mathbb{F}^2\mathbb{D}_q\mathbb{D}(\kappa_1\mathbb{F}\mathbb{D}_q - q\mathbb{Z}_1).$$

Thus, by Remark 4.2.1,

$$\mathcal{S}(\mathbb{T}_3 - \mathbb{F}\mathbb{D}_q\mathbb{Z}_2 + q\mathbb{F}^2\mathbb{D}_q\mathbb{D}\mathbb{Z}_1) = (\mathbb{Z}_2 - \mathbb{D}_q\mathbb{F})\mathbb{Z}_3 - \mathbb{F}\mathbb{D}_q(\kappa_1\mathbb{D}_q\mathbb{F} - q\mathbb{Z}_2) = \mathbb{0},$$

where the last equality follows from (5.6) and (5.9). It ends the proof of (5.19).
Now we will show (5.20). We use the definition of $\mathbb{T}_3$ and the fact that $\mathcal{Q}_0$ is commutative (in particular, $\mathbb{F}\mathbb{D}_q$ and $\mathbb{F}^2\mathbb{D}_q\mathbb{D}$ commute) to write

$$\begin{aligned}\mathbb{T}_3 - \mathbb{D}_q\mathbb{F}\mathbb{Z}_1 + q\mathbb{F}\mathbb{D}_q\mathbb{Z}_0 &= (\mathbb{Z}_2 - \mathbb{D}_q\mathbb{F})\mathbb{Z}_1 + \mathbb{F}\mathbb{D}_q(q\mathbb{Z}_0 - \kappa_1\mathbb{F}^2\mathbb{D}_q\mathbb{D})\\ &= -\mathbb{F}\mathbb{D}_q((q-\sigma t)\mathbb{Z}_1 - q\mathbb{Z}_0 + \kappa_1\mathbb{F}^2\mathbb{D}_q\mathbb{D}),\end{aligned}$$

where the last equality holds due to (5.6). In view of Remark 4.2.1, applying $\mathcal{S}^2$ to the above expression yields

$$\mathcal{S}^2(\mathbb{T}_3 - \mathbb{D}_q\mathbb{F}\mathbb{Z}_1 + q\mathbb{F}\mathbb{D}_q\mathbb{Z}_0) = -\mathcal{S}^2(\mathbb{F}\mathbb{D}_q)((q-\sigma t)\mathbb{Z}_3 - q\mathbb{Z}_2 + \kappa_1\mathbb{D}_q\mathbb{F}).$$

Formula (5.9) completes the proof of (5.20). □

5.1.5. Elements $\mathbb{K}_i$, $i = 1, 2, 3$

Using the notation given in (2.19), we define an auxiliary element

$$\mathbb{V} := \eta\kappa_0\mathcal{T}(\mathbb{F}\mathbb{D}_q) + \kappa_2\mathbb{Z}_1,$$

which is in $\mathcal{Q}_0$. The elements $\mathbb{Z}_1$ and $\mathbb{Z}_2$ commute as the elements of $\mathcal{Q}_0$, so according to (5.10) and (5.13), we have

$$\mathcal{S}(\mathbb{V})\mathbb{Z}_1 = \eta\kappa_0\mathbb{F}\mathbb{Z} + \kappa_2\mathbb{Z}_2\mathbb{Z}_1 = \eta\kappa_0(\mathbb{F}^2\mathbb{D}_q\mathbb{D}\mathbb{Z}_2 + \mathbb{F}\mathbb{Q}) + \kappa_2\mathbb{Z}_1\mathbb{Z}_2 = \mathbb{V}\mathbb{Z}_2 + \eta\kappa_0\mathbb{F}\mathbb{Q}. \tag{5.21}$$

We will now use $\mathbb{V}$ to define two important elements:

$$\mathbb{K}_1 := \mathbb{T}_2^{-1}(z\mathbb{Q} + \mathcal{T}(\mathbb{D}_q)\mathbb{V}), \tag{5.22}$$

$$\mathbb{K}_2 := \sigma\mathbb{K}_1 + \eta\mathbb{D}. \tag{5.23}$$

Both $\mathbb{K}_1$ and $\mathbb{K}_2$ belong to $\mathcal{Q}_1$, see Remark 4.1.2. Moreover, the nth coordinate of $\mathbb{K}_1$ is equal to

$$\frac{zq^{n-1}+[n-1]_q\Big(\eta\kappa_0[n-1]_q+\kappa_2(1+\sigma t[n-1]_q)\Big)}{1+\sigma t[2n-2]_q}x^{n-1} = \gamma_{n-1}(z)x^{n-1}, \qquad n \in \mathbb{N}, \tag{5.24}$$

recall the definition (2.21) of γ_n. The 0th coordinate of $\mathbb{K}_1$ is zero.
Furthermore, we will prove the following:

Lemma 5.1.6. *The element* $\mathbb{K}_1$ *satisfies*

$$\mathbb{T}_2\mathbb{K}_1\mathbb{F}\mathbb{D}_q\mathbb{Z}_2 - \mathbb{F}\mathbb{D}_q\mathbb{Z}_2\mathcal{S}(\mathbb{T}_2\mathbb{K}_1) = z\mathcal{S}(\mathbb{T}_3)\mathbb{Q} - \eta\kappa_0\mathbb{Q}\mathbb{F}^2\mathbb{D}_q^2.$$

Proof. The definitions of $\mathbb{K}_1$ and $\mathbb{V}$ give

$$\mathcal{S}(\mathbb{T}_2\mathbb{K}_1) = z\mathcal{S}(\mathbb{Q}) + \mathbb{D}_q\mathcal{S}(\mathbb{V}) = z\mathcal{S}(\mathbb{Q}) + \mathbb{Z}_2^{-1}\mathbb{D}_q\mathcal{S}(\mathbb{V})\mathbb{Z}_1 = qz\mathbb{Q} + \mathbb{Z}_2^{-1}\mathbb{D}_q\mathbb{V}\mathbb{Z}_2 + \eta\kappa_0\mathbb{Z}_2^{-1}\mathbb{D}_q\mathbb{F}\mathbb{Q},$$

where in the penultimate equality we used Lemma 4.2.2 applied to $\mathbb{D}_q\mathcal{S}(\mathbb{V}) \in \mathcal{Q}_1$ and $\mathbb{Z}_1 \in \mathcal{Q}_0$. In the last equality, we used identities (5.11) and (5.21). Multiplying the above formula by $\mathbb{F}\mathbb{D}_q\mathbb{Z}_2$ from the left and using (5.19), we get

$$\begin{aligned}\mathbb{F}\mathbb{D}_q\mathbb{Z}_2\mathcal{S}(\mathbb{T}_2\mathbb{K}_1) &= z\mathcal{S}(q\mathbb{F}^2\mathbb{D}_q\mathbb{D}\mathbb{Z}_1)\mathbb{Q} + \mathbb{F}\mathbb{D}_q^2\mathbb{V}\mathbb{Z}_2 + \eta\kappa_0\mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathbb{Q} \\ &= z\mathcal{S}(\mathbb{F}\mathbb{D}_q\mathbb{Z}_2 - \mathbb{T}_3)\mathbb{Q} + \mathcal{T}(\mathbb{D}_q)\mathbb{F}\mathbb{D}_q\mathbb{V}\mathbb{Z}_2 + \eta\kappa_0\mathcal{S}(\mathbb{F}^2\mathbb{D}_q^2)\mathbb{Q}.\end{aligned}$$

Lemma 4.2.2 applied to $\mathbb{Q} \in \mathcal{Q}_1$ along with $\mathbb{F}\mathbb{D}_q\mathbb{Z}_2 \in \mathcal{Q}_0$ and $\mathbb{F}^2\mathbb{D}_q^2 \in \mathcal{Q}_0$, respectively, yields

$$\mathbb{F}\mathbb{D}_q\mathbb{Z}_2\mathcal{S}(\mathbb{T}_2\mathbb{K}_1) = z\mathbb{Q}\mathbb{F}\mathbb{D}_q\mathbb{Z}_2 - z\mathcal{S}(\mathbb{T}_3)\mathbb{Q} + \mathcal{T}(\mathbb{D}_q)\mathbb{F}\mathbb{D}_q\mathbb{V}\mathbb{Z}_2 + \eta\kappa_0\mathbb{Q}\mathbb{F}^2\mathbb{D}_q^2.$$

Since $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$ and $\mathbb{V} \in \mathcal{Q}_0$ commute, putting together the first and the third term on the right-hand side above into $\mathbb{K}_1$ (compare with (5.22)), completes the proof. □

Additionally, it turns out that the following special element of $\mathcal{Q}$ will play a key role in the sequel:

$$\mathbb{K}_3 := (\sigma\mathcal{S}(\mathbb{K}_1) + \mathbb{K}_2)\mathbb{T}_3^{-1}. \tag{5.25}$$

The element $\mathbb{K}_3$ is symmetric with respect to $\sigma\mathbb{K}_1$ and $\mathbb{K}_2$, that is, it satisfies the following equation:

$$\mathbb{K}_3 = (\mathcal{S}(\mathbb{K}_2 - \eta\mathbb{D}) + \mathbb{K}_2)\mathbb{T}_3^{-1} = (\mathcal{S}(\mathbb{K}_2) + \mathbb{K}_2 - \eta\mathbb{D})\mathbb{T}_3^{-1} = (\sigma\mathbb{K}_1 + \mathcal{S}(\mathbb{K}_2))\mathbb{T}_3^{-1}, \tag{5.26}$$

see (5.23).

It is worth noting that the element $\mathbb{K}_3$ belongs to $\mathcal{Q}_1$ by Remark 4.1.2. Moreover, using (5.23), (5.24) and (5.17), the nth coordinate of $\mathbb{K}_3$ is equal to

$$\tfrac{\sigma\gamma_n(z)+\gamma_{n-1}(z)+\eta}{1+\sigma t[2n-1]_q}x^{n-1}, \qquad n \in \mathbb{N}. \tag{5.27}$$

The 0th coordinate of $\mathbb{K}_3$ is zero.

Let us record the following lemma for further reference:

Lemma 5.1.7. *The following identity holds:*

$$\kappa_0\mathbb{F}\mathbb{D}_q[\mathbb{K}_3, \mathbb{F}\mathbb{D}_q]\mathbb{F}\mathbb{D}_q = \mathbb{F}\mathbb{D}_q\mathcal{S}(\mathbb{K}_1)\mathbb{Z}_2 - \mathbb{Z}_2\mathbb{K}_1\mathbb{F}\mathbb{D}_q + z\mathbb{Q}.$$

Proof. Formulas (5.15) and (5.16) imply that

$$\mathbb{T}_3\mathbb{Z}_2 - \mathbb{Z}_1\mathbb{T}_2 = (\mathbb{Z}_1\mathbb{Z}_2 - \kappa_1\mathbb{F}^2\mathbb{D}_q^2)\mathbb{Z}_2 - \mathbb{Z}_1(\mathbb{Z}_2^2 - \kappa_1(\mathbb{F}\mathbb{D}_q)^2) = \kappa_1\mathbb{F}\mathbb{D}_q(\mathbb{F}\mathbb{Z} - \mathbb{F}^2\mathbb{D}_q\mathbb{D}\mathbb{Z}_2),$$

where in the last step, as in the proof of Lemma 5.1.5, we have used the fact that $\mathcal{Q}_0$ is commutative, in particular, $\mathbb{F}\mathbb{D}_q$ and $\mathbb{F}^2\mathbb{D}_q\mathbb{D}$ commute. From (5.13), the expression in the parentheses is equal to $\mathbb{F}\mathbb{Q}$. Lemma 4.2.3 applied to $\mathbb{Q} \in \mathcal{Q}_1$ and $\mathbb{D}_q \in \mathcal{Q}_1$ yields

$$\mathbb{T}_3\mathbb{Z}_2 - \mathbb{Z}_1\mathbb{T}_2 = \kappa_1\mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{D}_q.$$

Analogously, we can show that

$$\mathbb{T}_1\mathbb{Z}_2 - \mathbb{T}_3\mathbb{Z}_1 = \kappa_1\mathbb{F}\mathbb{Q}\mathbb{F}^2\mathbb{D}_q\mathbb{D}.$$

Before we use the last two expressions, note that using (5.12) first and then applying (5.25) and (5.23) yields:

$$\begin{aligned}\kappa_0\mathbb{F}\mathbb{D}_q[\mathbb{K}_3, \mathbb{F}\mathbb{D}_q]\mathbb{F}\mathbb{D}_q &= \kappa_0\mathbb{F}\mathbb{D}_q\mathbb{K}_3\mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{D}_q \\ &= \kappa_1\mathbb{F}\mathbb{D}_q\mathbb{K}_1\mathbb{T}_3^{-1}\mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{D}_q + \kappa_1\mathbb{F}\mathbb{D}_q\mathcal{S}(\mathbb{K}_1)\mathbb{T}_3^{-1}\mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{D}_q + \eta\kappa_0\mathbb{F}\mathbb{D}_q\mathbb{D}\mathbb{T}_3^{-1}\mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{D}_q.\end{aligned}$$

Lemma 4.2.2 used twice (in the first term with $\mathbb{F}\mathbb{D}_q\mathbb{K}_1 \in \mathcal{Q}_1$ and $\mathbb{T}_3^{-1}\mathbb{F}\mathbb{Q} \in \mathcal{Q}_0$, and in the third term with $\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{Q} \in \mathcal{Q}_1$ and $\mathbb{F}^2\mathbb{D}_q\mathbb{D} \in \mathcal{Q}_0$) yields

$$\begin{aligned}\kappa_0\mathbb{F}\mathbb{D}_q[\mathbb{K}_3, \mathbb{F}\mathbb{D}_q]\mathbb{F}\mathbb{D}_q = \kappa_1\mathcal{S}(\mathbb{T}_3^{-1}\mathbb{F}\mathbb{Q}\mathbb{F}^2\mathbb{D}_q\mathbb{D})\mathbb{K}_1\mathbb{F}\mathbb{D}_q + \kappa_1\mathbb{F}\mathbb{D}_q\mathcal{S}(\mathbb{K}_1)\mathbb{T}_3^{-1}\mathbb{F}\mathbb{Q}\mathbb{F}\mathbb{D}_q \\ + \eta\kappa_0\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{Q}\mathbb{F}^2\mathbb{D}_q^2.\end{aligned}$$

Now we are in a position to use the two expressions derived earlier:

$$\begin{aligned}\kappa_0\mathbb{F}\mathbb{D}_q[\mathbb{K}_3, \mathbb{F}\mathbb{D}_q]\mathbb{F}\mathbb{D}_q = \mathcal{S}(\mathbb{T}_3^{-1}\mathbb{T}_1\mathbb{Z}_2 - \mathbb{Z}_1)\mathbb{K}_1\mathbb{F}\mathbb{D}_q + \mathbb{F}\mathbb{D}_q\mathcal{S}(\mathbb{K}_1)(\mathbb{Z}_2 - \mathbb{T}_3^{-1}\mathbb{Z}_1\mathbb{T}_2) \\ + \eta\kappa_0\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{Q}\mathbb{F}^2\mathbb{D}_q^2.\end{aligned}$$

Lemma 4.2.2 yields that $\mathcal{S}(\mathbb{Z}_2)\mathbb{K}_1\mathbb{F}\mathbb{D}_q = \mathbb{K}_1\mathbb{F}\mathbb{D}_q\mathbb{Z}_2$ and $\mathcal{S}(\mathbb{K}_1)\mathbb{T}_3^{-1}\mathbb{Z}_1\mathbb{T}_2 = \mathcal{S}(\mathbb{T}_3^{-1}\mathbb{Z}_1\mathbb{T}_2\mathbb{K}_1)$. Furthermore, the commutativity of $\mathcal{S}(\mathbb{T}_3^{-1}) \in \mathcal{Q}_0$ and $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$, and the first equality in

(5.15) imply

$$\begin{aligned}\kappa_0\mathbb{F}\mathbb{D}_q[\mathbb{K}_3,\mathbb{F}\mathbb{D}_q]\mathbb{F}\mathbb{D}_q =&\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{T}_2\mathbb{K}_1\mathbb{F}\mathbb{D}_q\mathbb{Z}_2 - \mathcal{S}(\mathbb{T}_3^{-1})\mathbb{F}\mathbb{D}_q\mathbb{Z}_2\mathcal{S}(\mathbb{T}_2\mathbb{K}_1) + \eta\kappa_0\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{Q}\mathbb{F}^2\mathbb{D}_q^2\\ &+ \mathbb{F}\mathbb{D}_q\mathcal{S}(\mathbb{K}_1)\mathbb{Z}_2 - \mathbb{Z}_2\mathbb{K}_1\mathbb{F}\mathbb{D}_q.\end{aligned}$$

The assertion of Lemma 5.1.7 follows from Lemma 5.1.6 and (4.3). □

5.1.6. Element $\mathbb{B}$

Finally, we introduce an element

$$\mathbb{B} := \kappa_0\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{D}_q\mathcal{R}(\mathbb{K}_1)\mathbb{D}_q\mathbb{T}_3^{-1},$$

where $\mathcal{R}$ is defined by (2.24). Note that (2.24) implies that $\mathcal{R}(\mathbb{K}_1) \in \mathcal{Q}_0$, so $\mathbb{B} \in \mathcal{Q}_2$ by Remark 4.1.2.

Moreover, in view of (5.17) and (5.24), the nth coordinate of $\mathbb{B}$ is equal to

$$\kappa_0[n]_q[n-1]_q\tfrac{1+\eta\gamma_{n-1}(z)+\sigma\gamma_{n-1}(z)^2}{(1+\sigma t[2n-1]_q)(1+\sigma t[2n-3]_q)}x^{n-2}, \qquad n \in \mathbb{N}_0, \tag{5.28}$$

interpreted as zero for $n \in \{0,1\}$.

Lemma 5.1.8. *The element* $\mathbb{B}$ *can be represented as*

$$\mathbb{B} = \kappa_0\mathbb{D}_q^2\mathcal{T}(\mathcal{R}(\mathbb{K}_1)\mathbb{T}_3^{-1})\mathbb{T}_3^{-1}.$$

Proof. Lemma 4.2.2 applied to $\mathbb{D}_q\mathcal{R}(\mathbb{K}_1) \in \mathcal{Q}_1$ and $\mathbb{T}_3^{-1} \in \mathcal{Q}_0$ yields

$$\mathbb{B} = \kappa_0\mathbb{D}_q\mathcal{R}(\mathbb{K}_1)\mathbb{T}_3^{-1}\mathbb{D}_q\mathbb{T}_3^{-1}.$$

Lemma 4.2.2, used this time with $\mathbb{D}_q \in \mathcal{Q}_1$ and $\mathcal{T}(\mathcal{R}(\mathbb{K}_1)\mathbb{T}_3^{-1}) \in \mathcal{Q}_0$, ends the proof. □

5.2. More elements of $\mathcal{Q}$ and the representation of $\mathbb{U}$ and $\mathbb{Y}$

In the current section, we will focus on more complicated elements of $\mathcal{Q}$, derived from the building blocks introduced in the previous section. We will analyze their properties, which are crucial for our future considerations.

5.2.1. Element $\widetilde{\mathbb{P}}$ with its relatives

The primary object of our interest is

$$\widetilde{\mathbb{P}} := \widetilde{\mathbb{P}}_1 + \widetilde{\mathbb{P}}_2 + \widetilde{\mathbb{P}}_3$$

with

$$\widetilde{\mathbb{P}}_1 := \mathbb{E}, \qquad \widetilde{\mathbb{P}}_2 := \kappa_0 \mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathbb{K}_3, \qquad \widetilde{\mathbb{P}}_3 := \kappa_1 \mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathbb{B}.$$

In addition, we will consider some elements closely related to $\widetilde{\mathbb{P}}$. Namely, let us define

$$\breve{\mathbb{P}} := \breve{\mathbb{P}}_1 + \breve{\mathbb{P}}_2 + \breve{\mathbb{P}}_3 \qquad \text{and} \qquad \mathbb{P} := \mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3$$

with

$$\breve{\mathbb{P}}_1 := \mathbb{E}, \qquad \breve{\mathbb{P}}_2 := \kappa_0 \mathbb{D}_q\mathbb{F}\mathbb{K}_3\mathbb{F}\mathbb{D}_q, \qquad \breve{\mathbb{P}}_3 := \kappa_1 \mathbb{D}_q\mathbb{F}\mathbb{B}\mathbb{F}\mathbb{D}_q,$$

$$\mathbb{P}_1 := \mathbb{E}, \qquad \mathbb{P}_2 := \kappa_0 \mathbb{F}\mathbb{D}_q\mathbb{K}_3\mathbb{D}_q\mathbb{F}, \qquad \mathbb{P}_3 := \kappa_1 \mathbb{F}\mathbb{D}_q\mathbb{B}\mathbb{D}_q\mathbb{F}.$$

Remark 4.1.2 states that all elements $\widetilde{\mathbb{P}}_i$, $\breve{\mathbb{P}}_i$, $\mathbb{P}_i$ belong to $\mathcal{Q}_{i-1}$, $i = 1, 2, 3$. Moreover, Remark 4.1.5 shows that $\widetilde{\mathbb{P}}$, $\breve{\mathbb{P}}$, and $\mathbb{P}$ are invertible.

By the commutativity of $\mathcal{Q}_0$ (in particular, by the fact that $\mathbb{F}\mathbb{D}_q$ and $\mathbb{D}_q\mathbb{F}$ commute), we have:

$$\mathbb{F}\mathbb{D}_q\breve{\mathbb{P}} = \widetilde{\mathbb{P}}\mathbb{F}\mathbb{D}_q \qquad \text{and} \qquad \widetilde{\mathbb{P}}\mathbb{D}_q\mathbb{F} = \mathbb{D}_q\mathbb{F}\mathbb{P}. \tag{5.29}$$

Furthermore, in view of (2.4),

$$(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{P} = \mathbb{E} - \mathbb{F}\mathbb{D}. \tag{5.30}$$

Now we are ready to introduce the elements $\mathbb{U}$ and $\mathbb{Y}$ whose existence was postulated in Assumption A1. We define them as:

$$\mathbb{U} := \mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{P}}) \qquad \text{and} \qquad \mathbb{Y} := \mathbb{Z}_2\breve{\mathbb{P}}\mathbb{Z}_1. \tag{5.31}$$

The elements $\mathbb{U}$ and $\mathbb{Y}$ are invertible as products of invertible elements (recall (4.4)).

By the following lemma, $\mathbb{U}$ and $\mathbb{Y}$ defined by (5.31) confirm the validity of Assumption A1.

Lemma 5.2.1. *The elements* $\mathbb{U}$ *and* $\mathbb{Y}$ *satisfy*

$$\mathbb{D}_q\mathbb{Y} = \mathbb{U}\mathbb{Z}.$$

Proof. Note that $\mathbb{D}_q\mathbb{Z}_2\check{\mathbb{P}}$ is a sum of certain elements of $\mathcal{Q}_k$, $k = 1, 2, 3$, so Lemma 4.1.3 gives that $\mathbb{D}_q\mathbb{Z}_2\check{\mathbb{P}}\mathbb{F}\mathbb{D} = \mathbb{D}_q\mathbb{Z}_2\check{\mathbb{P}}$. Consequently,

$$\mathbb{D}_q\mathbb{Y} = \mathbb{D}_q\mathbb{Z}_2\check{\mathbb{P}}\mathbb{Z}_1 = \mathbb{D}_q\mathbb{Z}_2\check{\mathbb{P}}\mathbb{F}\mathbb{D}\mathbb{Z}_1 = \mathcal{S}(\mathbb{F}\mathbb{D}_q\mathbb{Z}_2\check{\mathbb{P}})\mathbb{D}\mathbb{Z}_1.$$

Commutation of $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$ and $\mathbb{Z}_2 \in \mathcal{Q}_0$, combined with the first identity in (5.29), leads to

$$\mathbb{D}_q\mathbb{Y} = \mathcal{S}(\mathbb{Z}_2\mathbb{F}\mathbb{D}_q\check{\mathbb{P}})\mathbb{D}\mathbb{Z}_1 = \mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{P}}\mathbb{F}\mathbb{D}_q)\mathbb{D}\mathbb{Z}_1 = \mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{P}})\mathbb{D}_q\mathbb{F}\mathbb{D}\mathbb{Z}_1 = \mathbb{U}\mathbb{D}_q\mathbb{F}\mathbb{D}\mathbb{Z}_1.$$

Lemma 4.1.3 applied to $\mathbb{D}_q \in \mathcal{Q}_1$ implies that $\mathbb{D}_q\mathbb{F}\mathbb{D}\mathbb{Z}_1 = \mathbb{D}_q\mathbb{Z}_1 = \mathbb{Z}$, see (5.10). □

We record the identity given by the following lemma for further reference:

Lemma 5.2.2. *The elements* $\mathbb{P}$ *and* $\mathbb{U}$ *satisfy*

$$\mathcal{S}(\mathbb{Z})\mathbb{P} = \mathbb{U}\mathcal{S}(\mathbb{D}_q).$$

Proof. From (5.29) and Remark 4.2.1, we get

$$\mathcal{S}(\mathbb{Z})\mathbb{P} = \mathbb{D}\mathbb{Z}_2\mathbb{D}_q\mathbb{F}\mathbb{P} = \mathbb{D}\mathbb{Z}_2\widetilde{\mathbb{P}}\mathbb{D}_q\mathbb{F} = \mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{P}}\mathbb{D}_q) = \mathbb{U}\mathcal{S}(\mathbb{D}_q).$$

□

5.2.2. Element $\mathbb{X}$ with its relatives

We will introduce new elements $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ to concisely describe the relation between $\widetilde{\mathbb{P}}$ and $\mathcal{S}(\widetilde{\mathbb{P}})$. Set:

$$\mathbb{X} := \mathbb{X}_2 + \mathbb{X}_3 \qquad \text{and} \qquad \widetilde{\mathbb{X}} := \widetilde{\mathbb{X}}_2 + \widetilde{\mathbb{X}}_3$$

with

$$\begin{aligned} &\mathbb{X}_2 := \mathcal{S}(\mathbb{K}_1), && \mathbb{X}_3 := \mathcal{S}(\mathbb{T}_3)\mathbb{B}, \\ &\widetilde{\mathbb{X}}_2 := \mathbb{K}_1, && \widetilde{\mathbb{X}}_3 := \kappa_0\mathcal{T}(\mathbb{D}_q^2\mathcal{R}(\mathbb{K}_1)\mathbb{T}_3^{-1}). \end{aligned} \tag{5.32}$$

Remark 4.1.2 shows that all elements $\mathbb{X}_i$ and $\widetilde{\mathbb{X}}_i$ indexed by i belong to $\mathcal{Q}_{i-1}$, $i = 2, 3$. Now, let us discuss some properties of these elements.

Lemma 5.2.3. *The following identities are satisfied:*

$$\begin{aligned}
\mathbb{X}_3 =& \kappa_0 \mathbb{D}_q \mathcal{R}(\widetilde{\mathbb{X}}_2) \mathbb{D}_q \mathbb{T}_3^{-1}, \\
\mathcal{S}(\widetilde{\mathbb{X}}_3) =& \kappa_0 \mathbb{D}_q \mathcal{R}(\mathbb{X}_2) \mathbb{D}_q \mathbb{T}_3^{-1}, \\
\mathcal{R}(\widetilde{\mathbb{X}}_2) =& \mathcal{R}(\mathbb{X}_2) + (\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\mathbb{F}\mathbb{K}_3\mathbb{T}_3\mathbb{F}.
\end{aligned}$$

Proof. The first identity is a direct consequence of (5.32), the definition of $\mathbb{B}$, and identity (4.3). Additionally, applying Lemma 4.2.2 to $\mathbb{D}_q \in \mathcal{Q}_1$ and $\mathcal{R}(\mathbb{K}_1) \in \mathcal{Q}_0$ we get

$$\mathcal{S}(\widetilde{\mathbb{X}}_3) = \kappa_0 \mathbb{D}_q^2 \mathcal{R}(\mathbb{K}_1) \mathbb{T}_3^{-1} = \kappa_0 \mathbb{D}_q \mathcal{S}(\mathcal{R}(\mathbb{K}_1)) \mathbb{D}_q \mathbb{T}_3^{-1}.$$

Lemma 4.2.4 completes the proof of the second identity. To prove the third formula, note that $\mathbb{X}_2\mathbb{F} \in \mathcal{Q}_0$ and $\widetilde{\mathbb{X}}_2\mathbb{F} \in \mathcal{Q}_0$, so they commute. Consequently,

$$(\widetilde{\mathbb{X}}_2\mathbb{F})^2 - (\mathbb{X}_2\mathbb{F})^2 = (\widetilde{\mathbb{X}}_2\mathbb{F} - \mathbb{X}_2\mathbb{F})(\widetilde{\mathbb{X}}_2\mathbb{F} + \mathbb{X}_2\mathbb{F}) = (\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\mathbb{F}(\widetilde{\mathbb{X}}_2 + \mathbb{X}_2)\mathbb{F}.$$

By the definition of $\mathcal{R}$ given by (2.24), we obtain

$$\mathcal{R}(\widetilde{\mathbb{X}}_2) - \mathcal{R}(\mathbb{X}_2) = \eta(\widetilde{\mathbb{X}}_2\mathbb{F} - \mathbb{X}_2\mathbb{F}) + \sigma((\widetilde{\mathbb{X}}_2\mathbb{F})^2 - (\mathbb{X}_2\mathbb{F})^2) = (\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\mathbb{F}(\eta\mathbb{D} + \sigma\widetilde{\mathbb{X}}_2 + \sigma\mathbb{X}_2)\mathbb{F}.$$

The third identity follows from the definitions of $\mathbb{X}_2$ and $\widetilde{\mathbb{X}}_2$, along with formulas (5.25) and (5.23). $\square$

5.2.3. Relations between $\widetilde{\mathbb{P}}$, $\mathbb{X}$, and their relatives

This subsection discusses some relations between the objects introduced in the previous sections.

Referring to the definition of $\mathbb{X}_3$, it is straightforward to observe that (4.3) leads to

$$\breve{\mathbb{P}}_3 = \kappa_1 \mathcal{S}(\mathbb{F}\mathbb{D}_q\mathbb{T}_3^{-1})\mathbb{X}_3\mathbb{F}\mathbb{D}_q, \qquad \widetilde{\mathbb{P}}_3 = \kappa_1 \mathcal{S}(\mathbb{F}^2\mathbb{D}_q^2\mathbb{T}_3^{-1})\mathbb{X}_3. \tag{5.33}$$

Moreover, the commutativity of $\mathcal{T}(\mathcal{R}(\mathbb{K}_1)\mathbb{T}_3^{-1})\mathbb{T}_3^{-1} \in \mathcal{Q}_0$ and $\mathbb{F}^2\mathbb{D}_q^2 \in \mathcal{Q}_0$ yields

$$\kappa_1\widetilde{\mathbb{X}}_3\mathbb{T}_3^{-1}\mathbb{F}^2\mathbb{D}_q^2 = \kappa_1\kappa_0\mathcal{T}(\mathbb{D}_q^2)\mathbb{F}^2\mathbb{D}_q^2\mathcal{T}(\mathcal{R}(\mathbb{K}_1)\mathbb{T}_3^{-1})\mathbb{T}_3^{-1} = \widetilde{\mathbb{P}}_3, \tag{5.34}$$

where the last step holds due to Lemma 5.1.8 and the definition of $\mathcal{T}$. Furthermore, the definitions of $\widetilde{\mathbb{X}}_3$ and $\mathcal{T}$ give that

$$\mathcal{S}(\mathbb{T}_3)\widetilde{\mathbb{X}}_3\mathbb{F}\mathbb{D}_q = \kappa_0\mathcal{S}(\mathbb{T}_3)\mathbb{F}\mathbb{D}_q^2\mathcal{R}(\mathbb{K}_1)\mathbb{T}_3^{-1}\mathbb{D}_q.$$

Lemma 4.2.2 used with $\mathbb{F}\mathbb{D}_q^2\mathcal{R}(\mathbb{K}_1) \in \mathcal{Q}_1$ and $\mathbb{T}_3^{-1} \in \mathcal{Q}_0$, together with identity (4.3) imply:

$$\mathcal{S}(\mathbb{T}_3)\widetilde{\mathbb{X}}_3\mathbb{F}\mathbb{D}_q = \kappa_0\mathbb{F}\mathbb{D}_q^2\mathcal{R}(\mathbb{K}_1)\mathbb{D}_q = \mathbb{F}\mathbb{D}_q\mathbb{X}_3\mathbb{T}_3, \tag{5.35}$$

where the last step follows from the definitions of $\mathbb{B}$ and $\mathbb{X}_3$.

Moreover, the elements $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ are related to $\widetilde{\mathbb{P}}$ as follows:

Lemma 5.2.4. *The following identity is satisfied:*

$$[\widetilde{\mathbb{P}}, \mathbb{F}\mathbb{D}_q] = \mathbb{F}\mathbb{D}_q\mathbb{X}\mathbb{Z}_2 - \mathbb{Z}_2\widetilde{\mathbb{X}}\mathbb{F}\mathbb{D}_q + z\mathbb{Q}.$$

Proof. Since the elements $\widetilde{\mathbb{P}}$, $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ are certain linear combinations of some elements from $\mathcal{Q}_k$, $k = 0, 1, 2$, the asserted equality will be proved by comparing the summands from these subspaces.

1. Elements of $\mathcal{Q}_0$. Restricted to $\mathcal{Q}_0$, the identity from Lemma 5.2.4 states that $[\widetilde{\mathbb{P}}_1, \mathbb{F}\mathbb{D}_q] = \mathbb{0}$, which is implied by Remark 4.2.7.
2. Elements of $\mathcal{Q}_1$. When restricted to $\mathcal{Q}_1$, the asserted identity becomes

$$[\widetilde{\mathbb{P}}_2, \mathbb{F}\mathbb{D}_q] = \mathbb{F}\mathbb{D}_q\mathbb{X}_2\mathbb{Z}_2 - \mathbb{Z}_2\widetilde{\mathbb{X}}_2\mathbb{F}\mathbb{D}_q + z\mathbb{Q}.$$

   This can be proved in the following way. Lemma 4.2.2 applied to $\mathbb{K}_3 \in \mathcal{Q}_1$ and $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$ yields that $\widetilde{\mathbb{P}}_2 = \kappa_0\mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathbb{K}_3 = \kappa_0\mathbb{F}\mathbb{D}_q\mathbb{K}_3\mathbb{F}\mathbb{D}_q$. Next, in view of Lemma 4.2.8, we get

$$[\widetilde{\mathbb{P}}_2, \mathbb{F}\mathbb{D}_q] = \kappa_0\mathbb{F}\mathbb{D}_q[\mathbb{K}_3, \mathbb{F}\mathbb{D}_q]\mathbb{F}\mathbb{D}_q = \mathbb{F}\mathbb{D}_q\mathbb{X}_2\mathbb{Z}_2 - \mathbb{Z}_2\widetilde{\mathbb{X}}_2\mathbb{F}\mathbb{D}_q + z\mathbb{Q},$$

   where the last equality is satisfied because of Lemma 5.1.7.

3. Elements of $\mathcal{Q}_2$. Finally, Lemma 5.2.4 restricted to $\mathcal{Q}_2$ says

$$[\widetilde{\mathbb{P}}_3, \mathbb{F}\mathbb{D}_q] = \mathbb{F}\mathbb{D}_q \mathbb{X}_3 \mathbb{Z}_2 - \mathbb{Z}_2 \widetilde{\mathbb{X}}_3 \mathbb{F}\mathbb{D}_q. \tag{5.36}$$

The proof relies on the following arguments. Remark 4.2.9 applied to $\widetilde{\mathbb{P}}_3$, $\mathbb{F}\mathbb{D}_q$, and (5.5) for $k = 2$, gives

$$[\widetilde{\mathbb{P}}_3, \mathbb{F}\mathbb{D}_q] = \mathbb{Z}_2 \widetilde{\mathbb{P}}_3 \mathbb{F}\mathbb{D}_q - \mathbb{F}\mathbb{D}_q \widetilde{\mathbb{P}}_3 \mathbb{Z}_2.$$

Due to (5.34) and (5.33), we get

$$[\widetilde{\mathbb{P}}_3, \mathbb{F}\mathbb{D}_q] = \kappa_1 \mathbb{Z}_2 \widetilde{\mathbb{X}}_3 \mathbb{T}_3^{-1} \mathbb{F}^2 \mathbb{D}_q^2 \mathbb{F}\mathbb{D}_q - \kappa_1 \mathbb{F}\mathbb{D}_q \mathcal{S}(\mathbb{F}^2 \mathbb{D}_q^2 \mathbb{T}_3^{-1}) \mathbb{X}_3 \mathbb{Z}_2.$$

Formula (5.16) applied to $\kappa_1 \mathbb{F}^2 \mathbb{D}_q^2$ which appears twice above gives

$$[\widetilde{\mathbb{P}}_3, \mathbb{F}\mathbb{D}_q] = \mathbb{Z}_2 \widetilde{\mathbb{X}}_3 \mathbb{T}_3^{-1} \mathbb{Z}_1 \mathbb{Z}_2 \mathbb{F}\mathbb{D}_q - \mathbb{Z}_2 \widetilde{\mathbb{X}}_3 \mathbb{F}\mathbb{D}_q - \mathbb{F}\mathbb{D}_q \mathcal{S}(\mathbb{Z}_1 \mathbb{Z}_2 \mathbb{T}_3^{-1}) \mathbb{X}_3 \mathbb{Z}_2 + \mathbb{F}\mathbb{D}_q \mathbb{X}_3 \mathbb{Z}_2. \tag{5.37}$$

Since $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$ and $\mathbb{T}_3^{-1} \mathbb{Z}_1 \mathbb{Z}_2 \in \mathcal{Q}_0$ commute,

$$\mathbb{Z}_2 \widetilde{\mathbb{X}}_3 \mathbb{T}_3^{-1} \mathbb{Z}_1 \mathbb{Z}_2 \mathbb{F}\mathbb{D}_q = \mathbb{Z}_2 \widetilde{\mathbb{X}}_3 \mathbb{F}\mathbb{D}_q \mathbb{T}_3^{-1} \mathbb{Z}_1 \mathbb{Z}_2 = \mathbb{Z}_2 \mathcal{S}(\mathbb{T}_3^{-1}) \mathbb{F}\mathbb{D}_q \mathbb{X}_3 \mathbb{Z}_1 \mathbb{Z}_2,$$

see (5.35) and (4.3). Lemma 4.2.2 applied to $\mathcal{S}(\mathbb{T}_3^{-1}) \mathbb{F}\mathbb{D}_q \mathbb{X}_3 \in \mathcal{Q}_2$ and $\mathbb{Z}_1 \in \mathcal{Q}_0$ yields that $\mathbb{Z}_2 \mathcal{S}(\mathbb{T}_3^{-1}) \mathbb{F}\mathbb{D}_q \mathbb{X}_3 \mathbb{Z}_1 \mathbb{Z}_2 = \mathcal{S}(\mathbb{Z}_1 \mathbb{Z}_2 \mathbb{T}_3^{-1}) \mathbb{F}\mathbb{D}_q \mathbb{X}_3 \mathbb{Z}_2$. By the commutativity of $\mathcal{S}(\mathbb{Z}_1 \mathbb{Z}_2 \mathbb{T}_3^{-1}) \in \mathcal{Q}_0$ and $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$, the right-hand side of (5.37) simplifies to (5.36).

□

Moreover, we can change the order of multiplication of $\mathcal{S}(\widetilde{\mathbb{P}})$ and $\mathbb{X}$ as follows:

Lemma 5.2.5. *The following identity is satisfied:*

$$\mathcal{S}(\widetilde{\mathbb{X}}) \widetilde{\mathbb{P}} = \mathcal{S}(\widetilde{\mathbb{P}}) \mathbb{X}.$$

Proof. As in the proof of Lemma 5.2.4, we will prove the asserted identity for elements restricted to the subspaces $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, and $\mathcal{Q}_4$, respectively. Recall that the elements $\widetilde{\mathbb{P}}_i$, $\mathbb{X}_i$, $\widetilde{\mathbb{X}}_i$ belong to $\mathcal{Q}_{i-1}$, and that a product of two elements belongs to the corresponding subspace as described in Remark 4.1.2.

1. Elements of $\mathcal{Q}_1$. We will show that $\mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_1 = \mathcal{S}(\mathbb{P}_1)\mathbb{X}_2$. In order to see this, it is enough to use the definitions of $\widetilde{\mathbb{P}}_1$, $\widetilde{\mathbb{X}}_2$, and $\mathbb{X}_2$.
2. Elements of $\mathcal{Q}_2$. When restricted to $\mathcal{Q}_2$, Lemma 5.2.5 says

$$\mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_2 + \mathcal{S}(\widetilde{\mathbb{X}}_3) = \mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_2 + \mathbb{X}_3. \tag{5.38}$$

We will prove it as follows. Lemma 5.2.3 gives

$$\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_3) = \kappa_0\mathbb{D}_q(\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\mathbb{F}\mathbb{K}_3\mathbb{T}_3\mathbb{F}\mathbb{D}_q\mathbb{T}_3^{-1} = \kappa_0\mathbb{D}_q(\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\mathbb{F}\mathbb{K}_3\mathbb{F}\mathbb{D}_q,$$

where the last step is satisfied since $\mathbb{T}_3 \in \mathcal{Q}_0$ and $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$ commute. Lemma 4.2.6 applied to $\mathbb{D}_q \in \mathcal{Q}_1$ and $\widetilde{\mathbb{X}}_2 - \mathbb{X}_2 \in \mathcal{Q}_1$ yields

$$\mathbb{D}_q(\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\mathbb{F} = \mathcal{S}(\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\mathbb{F}\mathbb{D}_q$$

whereas Lemma 4.2.3 used with $\mathbb{K}_3 \in \mathcal{Q}_1$ and $\mathbb{D}_q \in \mathcal{Q}_1$ implies that $\mathbb{K}_3\mathbb{F}\mathbb{D}_q = \mathbb{D}_q\mathbb{F}\mathbb{K}_3$. Consequently, $\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_3) = \mathcal{S}(\widetilde{\mathbb{X}}_2 - \mathbb{X}_2)\widetilde{\mathbb{P}}_2$. Lemma 4.2.6 (used with $\mathcal{S}(\mathbb{X}_2) \in \mathcal{Q}_1$ and $\widetilde{\mathbb{P}}_2 \in \mathcal{Q}_1$) and Lemma 4.2.5 yield (5.38).
3. Elements of $\mathcal{Q}_3$. Here our task is to prove

$$\mathcal{S}(\widetilde{\mathbb{X}}_3)\widetilde{\mathbb{P}}_2 + \mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_3 = \mathcal{S}(\widetilde{\mathbb{P}}_3)\mathbb{X}_2 + \mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_3. \tag{5.39}$$

In order to do so, we proceed as follows. Lemma 4.2.2 applied to $\widetilde{\mathbb{X}}_2 \in \mathcal{Q}_1$ and $\mathbb{F}^2\mathbb{D}_q^2 \in \mathcal{Q}_0$ yields that $\widetilde{\mathbb{X}}_2\mathbb{F}^2\mathbb{D}_q^2 = \mathbb{F}\mathbb{D}_q^2\mathbb{F}\widetilde{\mathbb{X}}_2$. Consequently, the second identity from (5.33) implies

$$\mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_3 = \mathcal{S}(\widetilde{\mathbb{P}}_2 - \kappa_1\mathbb{F}\mathbb{D}_q^2\mathbb{F}\widetilde{\mathbb{X}}_2\mathbb{T}_3^{-1})\mathbb{X}_3 = \kappa_0\mathcal{S}(\mathbb{F}\mathbb{D}_q^2\mathbb{F}(\mathbb{K}_3 - \sigma\mathbb{K}_1\mathbb{T}_3^{-1}))\mathbb{X}_3,$$

where in the last equality we have used the definitions of $\widetilde{\mathbb{P}}_2$ and $\widetilde{\mathbb{X}}_2$. Formula (5.26) gives $\mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_3 = \kappa_0\mathcal{S}(\mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathcal{S}(\mathbb{K}_2)\mathbb{T}_3^{-1})\mathbb{X}_3$. Remark 4.2.1 and Lemma 4.2.2 (applied to $\mathcal{S}^2(\mathbb{F}\mathbb{D}_q\mathbb{K}_2)\mathcal{S}(\mathbb{T}_3^{-1}) \in \mathcal{Q}_1$ and $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$) give

$$\mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_3 = \kappa_0\mathcal{S}^2(\mathbb{F}\mathbb{D}_q\mathbb{K}_2)\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{F}\mathbb{D}_q\mathbb{X}_3.$$

In view of (5.35),

$$\mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_3 = \kappa_0 \mathcal{S}^2(\mathbb{F}\mathbb{D}_q\mathbb{K}_2)\widetilde{\mathbb{X}}_3\mathbb{F}\mathbb{D}_q\mathbb{T}_3^{-1}.$$

Lemma 4.2.6 (applied to $\mathcal{S}(\widetilde{\mathbb{X}}_3\mathbb{F}\mathbb{D}_q) \in \mathcal{Q}_2$ and $\mathbb{F}\mathbb{D}_q\mathbb{K}_2 \in \mathcal{Q}_1$) and Lemma 4.2.5 yield

$$\mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_3 = \kappa_0 \mathcal{S}(\widetilde{\mathbb{X}}_3\mathbb{F}\mathbb{D}_q)\mathbb{F}\mathbb{D}_q\mathbb{K}_2\mathbb{T}_3^{-1}.$$

Note that the elements $\mathbb{F}\mathbb{D}_q \in \mathcal{Q}_0$ and $\mathbb{D}_q\mathbb{F} \in \mathcal{Q}_0$ commute. Moreover, (5.26) implies

$$\mathcal{S}(\widetilde{\mathbb{P}}_2)\mathbb{X}_3 - \mathcal{S}(\widetilde{\mathbb{X}}_2)\widetilde{\mathbb{P}}_3 = \kappa_0 \mathcal{S}(\widetilde{\mathbb{X}}_3)\mathbb{F}\mathbb{D}_q^2\mathbb{F}(\mathbb{K}_3 - \sigma\mathcal{S}(\mathbb{K}_1)\mathbb{T}_3^{-1}) = \mathcal{S}(\widetilde{\mathbb{X}}_3)\widetilde{\mathbb{P}}_2 - \kappa_1 \mathcal{S}(\widetilde{\mathbb{X}}_3\mathbb{T}_3^{-1}\mathbb{F}^2\mathbb{D}_q^2)\mathbb{X}_2.$$

Above we have used Lemma 4.2.2 again, this time applied to $\mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathcal{S}(\mathbb{K}_1) \in \mathcal{Q}_1$ and $\mathbb{T}_3^{-1} \in \mathcal{Q}_0$. Finally, formula (5.34) implies (5.39).

4. Elements of $\mathcal{Q}_4$. Finally, we have to show that $\mathcal{S}(\widetilde{\mathbb{P}}_3)\mathbb{X}_3 = \mathcal{S}(\widetilde{\mathbb{X}}_3)\widetilde{\mathbb{P}}_3$. Using the formula for $\widetilde{\mathbb{P}}_3$ given by (5.34) and the fact that $\mathbb{T}_3^{-1} \in \mathcal{Q}_0$ and $\mathbb{F}^2\mathbb{D}_q^2 \in \mathcal{Q}_0$ commute, we get

$$\mathcal{S}(\widetilde{\mathbb{P}}_3)\mathbb{X}_3 = \mathcal{S}(\kappa_1\widetilde{\mathbb{X}}_3\mathbb{T}_3^{-1}\mathbb{F}^2\mathbb{D}_q^2)\mathbb{X}_3 = \mathcal{S}(\kappa_1\widetilde{\mathbb{X}}_3\mathbb{F}^2\mathbb{D}_q^2\mathbb{T}_3^{-1})\mathbb{X}_3 = \mathcal{S}(\widetilde{\mathbb{X}}_3)\widetilde{\mathbb{P}}_3,$$

where the last equality comes from (5.33).

□

Chapter 6

Removing Assumptions A1, A2 and A3

In this chapter we will show that Assumptions A1-A3 are implied by (1.9) with $\sigma\tau = 0$. Our arguments will heavily rely on the results derived in Chapters 4 and 5. Therefore, as in Chapter 5, we will work under assumptions (1.9) with $\sigma\tau = 0$.

Recall that $z \in \mathbb{R}$ and $t \geqslant 0$ are fixed and all arguments (z, t) are suppressed, e.g. in $\mathbb{U}$, $\mathbb{Y}$.

6.1. Assumption A1

The elements $\mathbb{U}$ and $\mathbb{Y}$ (given by (5.31)) are invertible (see the line below (5.31)). The equality in Assumption A1 is an immediate consequence of Lemma 5.2.1.

6.2. Assumption A2

Before going to the proof of how Assumption A2 follows from (1.9) with $\sigma\tau = 0$, we will represent the element $\mathbb{S}$, given by (2.27), in terms of the objects presented in the previous chapter.

The following lemma verifies that $\widetilde{\mathbb{P}} + \mathbb{X}$ is equal to $\mathbb{S}$, which means that $\widetilde{\mathbb{P}} + \mathbb{X}$ encodes the Jacobi matrix for the orthogonal polynomials $\{W_n(\cdot; z, t)\}_{n=0}^{\infty}$ in $\mathcal{Q}$.

Lemma 6.2.1. *The following identity is satisfied:*

$$\mathbb{S} = \widetilde{\mathbb{P}} + \mathbb{X}. \tag{6.1}$$

Proof. Using the definitions of $\widetilde{\mathbb{P}}$ and $\mathbb{X}$, we can write

$$\widetilde{\mathbb{P}} + \mathbb{X} = \mathbb{E} + (\kappa_0 \mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathbb{K}_3 + \mathcal{S}(\mathbb{K}_1)) + \mathcal{S}(\kappa_1\mathbb{F}^2\mathbb{D}_q^2 + \mathbb{T}_3)\mathbb{B},$$

which, in view of (5.16), can be simplified to

$$\widetilde{\mathbb{P}} + \mathbb{X} = \mathbb{E} + (\kappa_0 \mathbb{F}\mathbb{D}_q^2\mathbb{F}\mathbb{K}_3 + \mathcal{S}(\mathbb{K}_1)) + \mathbb{Z}_2\mathbb{Z}_3\mathbb{B}.$$

Using (5.24), (5.27), and (5.28), we get that the nth coordinate of $\widetilde{\mathbb{P}}+\mathbb{X}$ is equal to (2.27), $n \in \mathbb{N}_0$. Since $\mathbb{S}$ and $\widetilde{\mathbb{P}} + \mathbb{X}$ coincide coordinate-wise, we get the desired result. □

Recall that $\widetilde{\mathbb{P}}$ and $\mathbb{X}$ are certain sums of elements of $\mathcal{Q}_k$, $k = 0, 1, 2$. By (6.1), the same is true for $\mathbb{S}$, so Lemma 4.1.3 implies

$$\mathbb{S}(\mathbb{E} - \mathbb{F}\mathbb{D}) = \widetilde{\mathbb{P}}_1(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{E} - \mathbb{F}\mathbb{D}. \tag{6.2}$$

We are now ready to present a result justifying that $\mathbb{S}$ is indeed related to the solution of the q-commutation equation, compare with Assumption A2.

Theorem 6.2.2. *The following identity is satisfied:*

$$\begin{aligned}(1+\sigma t)\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{Y} = (q-\sigma t)\mathbb{U}\mathbb{S}\mathbb{F}\mathbb{Z} + \mathbb{U}\mathbb{Y} + \kappa_0\mathbb{D}_q(\mathcal{R}(\mathbb{S}) - \mathcal{R}(\mathbb{X}))\mathbb{Z}&\\ + (\kappa_2\mathbb{D}_q\mathbb{Z}_2 + \mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3))\check{\mathbb{P}}\mathbb{Z}_1,&\end{aligned} \tag{6.3}$$

where $\mathbb{U}$ and $\mathbb{Y}$ are defined in (5.31)*.*

Proof. It is sufficient to prove each of the following four identities:

$$\begin{aligned}(1+\sigma t)\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{Y} - \mathbb{U}\mathbb{Y} = \mathbb{D}_q\mathbb{X}\mathbb{F}\mathbb{Z}_3\check{\mathbb{P}}\mathbb{Z}_1 + \kappa_1\mathbb{D}_q\mathbb{X}\mathbb{F}\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z} + q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z}&\\ - q[\mathcal{S}(\widetilde{\mathbb{P}}), \mathbb{F}\mathbb{D}_q]\check{\mathbb{P}}\mathbb{Z}_1,&\end{aligned} \tag{6.4}$$

$$\begin{aligned}(q-\sigma t)\mathbb{U}\mathbb{S}\mathbb{F}\mathbb{Z} + \kappa_0\mathbb{D}_q(\mathcal{R}(\mathbb{S}) - \mathcal{R}(\mathbb{X}))\mathbb{Z} = q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z} + q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{X}})\mathbb{F}\mathbb{D}_q\check{\mathbb{P}}\mathbb{Z}_1&\\ + \kappa_1\mathbb{D}_q\mathbb{X}\mathbb{F}\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z} + \eta\kappa_0\mathbb{D}_q\mathbb{F}\mathbb{D}_q\check{\mathbb{P}}\mathbb{Z}_1,&\end{aligned} \tag{6.5}$$

$$\mathbb{D}_q\mathbb{X}\mathbb{F}\mathbb{Z}_3 - q[\mathcal{S}(\widetilde{\mathbb{P}}), \mathbb{F}\mathbb{D}_q] = \mathbb{Z}_3\mathcal{S}(\widetilde{\mathbb{X}})\mathbb{D}_q\mathbb{F} - z\mathcal{S}(\mathbb{Q}), \tag{6.6}$$

$$\mathbb{Z}_3\mathcal{S}(\widetilde{\mathbb{X}})\mathbb{D}_q\mathbb{F} - q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{X}})\mathbb{F}\mathbb{D}_q = z\mathcal{S}(\mathbb{Q}) + \eta\kappa_0\mathbb{D}_q\mathbb{F}\mathbb{D}_q + \kappa_2\mathbb{D}_q\mathbb{Z}_2 + \mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3), \tag{6.7}$$

as they imply the desired result after obvious calculations.

1. Proof of (6.4). Remark 4.2.1 implies that $\mathbb{D}_q\mathbb{F}\mathcal{S}(\widetilde{\mathbb{P}}) = \mathcal{S}(\mathbb{F}\mathbb{D}_q\widetilde{\mathbb{P}})$. Moreover, identities (5.8) and (6.1) yield that

$$\begin{aligned}\mathbb{U}\mathbb{Y} = \mathbb{Z}_3\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{Y} &= ((1+\sigma t)\mathbb{D}_q\mathbb{F} - q\mathbb{F}\mathbb{D}_q)\,\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{Y} \\ &= (1+\sigma t)\mathcal{S}(\mathbb{F}\mathbb{D}_q(\mathbb{S}-\mathbb{X}))\mathbb{Y} - q\mathbb{F}\mathbb{D}_q\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{Y} \\ &= (1+\sigma t)\mathbb{D}_q(\mathbb{S}-\mathbb{X})\mathbb{F}\mathbb{Y} - q\mathbb{F}\mathbb{D}_q\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{Y}.\end{aligned}$$

   The definition of $\mathbb{Y}$ and formula (5.7) lead to

$$\begin{aligned}\mathbb{U}\mathbb{Y} &= (1+\sigma t)\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{Y} - \mathbb{D}_q\mathbb{X}\mathbb{F}\mathbb{Z}_3\check{\mathbb{P}}\mathbb{Z}_1 - \kappa_1\mathbb{D}_q\mathbb{X}\mathbb{F}^2\mathbb{D}_q\check{\mathbb{P}}\mathbb{Z}_1 - q\mathbb{F}\mathbb{D}_q\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{Z}_2\check{\mathbb{P}}\mathbb{Z}_1 \\ &= (1+\sigma t)\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{Y} - \mathbb{D}_q\mathbb{X}\mathbb{F}\mathbb{Z}_3\check{\mathbb{P}}\mathbb{Z}_1 - \kappa_1\mathbb{D}_q\mathbb{X}\mathbb{F}\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z} - q\mathbb{F}\mathbb{D}_q\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{Z}_2\check{\mathbb{P}}\mathbb{Z}_1,\end{aligned} \tag{6.8}$$

   where the last step follows from (5.29) and (5.10). Remark 4.2.9 applied to $\mathcal{S}(\widetilde{\mathbb{P}})$, $\mathbb{F}\mathbb{D}_q$, and $\mathbb{Z}_2$ yields

$$\mathbb{F}\mathbb{D}_q\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{Z}_2 = \mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{F}\mathbb{D}_q - [\mathcal{S}(\widetilde{\mathbb{P}}), \mathbb{F}\mathbb{D}_q].$$

   Substituting the above into (6.8) and using (5.29) again gives (6.4).

2. Proof of (6.5). Identity (5.9) together with the definition of $\mathbb{U}$ gives

$$(q-\sigma t)\mathbb{U}\mathbb{S}\mathbb{F}\mathbb{Z} = (q\mathbb{Z}_2 - \kappa_1\mathbb{D}_q\mathbb{F})\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{S}\mathbb{F}\mathbb{Z} = q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})\mathbb{S}\mathbb{F}\mathbb{Z} - \kappa_1\mathcal{S}(\mathbb{F}\mathbb{D}_q\widetilde{\mathbb{P}})\mathbb{S}\mathbb{F}\mathbb{Z}.$$

   Formula (6.1) implies

$$(q-\sigma t)\mathbb{U}\mathbb{S}\mathbb{F}\mathbb{Z} = q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})(\widetilde{\mathbb{P}}+\mathbb{X})\mathbb{F}\mathbb{Z} - \kappa_1\mathbb{D}_q(\mathbb{S}\mathbb{F})^2\mathbb{Z} + \kappa_1\mathbb{D}_q\mathbb{X}\mathbb{F}(\widetilde{\mathbb{P}}+\mathbb{X})\mathbb{F}\mathbb{Z}.$$

   Lemma 5.2.5 and then (5.29) with (5.10) yield that

$$q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})(\widetilde{\mathbb{P}}+\mathbb{X})\mathbb{F}\mathbb{Z} = q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z} + q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{X}})\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z} = q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{P}})\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z} + q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{X}})\mathbb{F}\mathbb{D}_q\check{\mathbb{P}}\mathbb{Z}_1.$$

Moreover, note that formulas (2.24), (5.2) and (6.1) give

$$\begin{aligned}\kappa_1\mathbb{D}_q(\mathbb{X}\mathbb{F})^2\mathbb{Z}-\kappa_1\mathbb{D}_q(\mathbb{S}\mathbb{F})^2\mathbb{Z}&=\kappa_0\mathbb{D}_q(\mathcal{R}(\mathbb{X})-\mathcal{R}(\mathbb{S}))\mathbb{Z}+\eta\kappa_0\mathbb{D}_q(\mathbb{S}-\mathbb{X})\mathbb{F}\mathbb{Z}\\&=\kappa_0\mathbb{D}_q(\mathcal{R}(\mathbb{X})-\mathcal{R}(\mathbb{S}))\mathbb{Z}+\eta\kappa_0\mathbb{D}_q\widetilde{\mathbb{P}}\mathbb{F}\mathbb{Z}\\&=\kappa_0\mathbb{D}_q(\mathcal{R}(\mathbb{X})-\mathcal{R}(\mathbb{S}))\mathbb{Z}+\eta\kappa_0\mathbb{D}_q\mathbb{F}\mathbb{D}_q\check{\mathbb{P}}\mathbb{Z}_1,\end{aligned}$$

where the last step is satisfied due to (5.29) and (5.10).

3. Proof of (6.6). Note that

$$q[\mathcal{S}(\widetilde{\mathbb{P}}),\mathbb{F}\mathbb{D}_q]=[\mathcal{S}(\widetilde{\mathbb{P}}),q\mathbb{F}\mathbb{D}_q]=[\mathcal{S}(\widetilde{\mathbb{P}}),\mathbb{D}_q\mathbb{F}-\mathbb{E}]=[\mathcal{S}(\widetilde{\mathbb{P}}),\mathbb{D}_q\mathbb{F}]=\mathcal{S}([\widetilde{\mathbb{P}},\mathbb{F}\mathbb{D}_q]),$$

where we used (4.5), (5.3), (4.7), (4.6) in each consecutive equality, respectively. Lemma 5.2.4, used with Remark 4.2.1 and the fact that $\mathbb{Z}_3=\mathcal{S}(\mathbb{Z}_2)$, ends the proof of (6.6).

4. Proof of (6.7). Note that Lemma 4.2.2 implies that

$$\mathbb{Z}_3\mathcal{S}(\widetilde{\mathbb{X}}_i)\mathbb{D}_q\mathbb{F}=\mathcal{S}^{i-1}(\mathbb{D}_q\mathbb{F})\mathbb{Z}_3\mathcal{S}(\widetilde{\mathbb{X}}_i),\qquad \mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{X}}_i)\mathbb{F}\mathbb{D}_q=\mathcal{S}^{i-1}(\mathbb{F}\mathbb{D}_q)\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{X}}_i),\qquad i=2,3.$$

Moreover, the identities $\mathbb{Z}_i=\mathcal{S}(\mathbb{Z}_{i-1})$, $i=1,2,3$, and (5.10) yield

$$\begin{aligned}\mathbb{Z}_3\mathcal{S}(\widetilde{\mathbb{X}})\mathbb{D}_q\mathbb{F}-q\mathbb{Z}_2\mathcal{S}(\widetilde{\mathbb{X}})\mathbb{F}\mathbb{D}_q&=\mathcal{S}(\mathbb{D}_q\mathbb{F}\mathbb{Z}_2-q\mathbb{F}\mathbb{Z})\mathcal{S}(\widetilde{\mathbb{X}}_2)+\mathcal{S}^2(\mathbb{D}_q\mathbb{F}\mathbb{Z}_1-q\mathbb{F}\mathbb{D}_q\mathbb{Z}_0)\mathcal{S}(\widetilde{\mathbb{X}}_3)\\&=\mathcal{S}(\mathbb{T}_2\mathbb{K}_1)+\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3),\end{aligned}$$

see (5.18), (5.32), and (5.20). In view of (5.22) and the definition of $\mathbb{V}$,

$$\mathcal{S}(\mathbb{T}_2\mathbb{K}_1)=z\mathcal{S}(\mathbb{Q})+\eta\kappa_0\mathbb{D}_q\mathbb{F}\mathbb{D}_q+\kappa_2\mathbb{D}_q\mathbb{Z}_2.$$

□

Expression (6.3) derived in Theorem 6.2.2 is almost identical to that in Assumption A2, see (5.31). Fortunately, the terms in (6.3) that do not appear in Assumption A2 cancel out:

Lemma 6.2.3. *The following identity is satisfied:*

$$\kappa_0\mathbb{D}_q\mathcal{R}(\mathbb{X})\mathbb{Z}=\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3)\check{\mathbb{P}}\mathbb{Z}_1.$$

Proof. Note that Lemma 4.2.2 used with $\mathbb{K}_3 \in \mathcal{Q}_1$ and $\mathbb{T}_3 \in \mathcal{Q}_0$ yields that $\mathcal{S}(\mathbb{T}_3^{-1})\mathbb{K}_3\mathbb{T}_3 = \mathbb{K}_3$. Next, referring to the definitions of $\breve{\mathbb{P}}_2$ and $\breve{\mathbb{P}}_3$, and to formula (5.2) we get

$$\begin{aligned}\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3)(\breve{\mathbb{P}}_2+\breve{\mathbb{P}}_3)\mathbb{Z}_1 &= \kappa_0\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3\mathbb{F}\mathbb{D}_q\mathbb{T}_3^{-1})(\mathbb{K}_3\mathbb{T}_3+\sigma\mathcal{S}(\mathbb{T}_3)\mathbb{B})\mathbb{F}\mathbb{Z}\\ &= \kappa_0\mathcal{S}(\mathbb{F}\mathbb{D}_q\mathbb{X}_3)(\mathbb{K}_3\mathbb{T}_3+\sigma\mathbb{X}_3)\mathbb{F}\mathbb{Z},\end{aligned}$$

where the last step holds true due to (5.35) and the definition of $\mathbb{X}_3$. Identities (5.25) and (5.23) yield

$$\begin{aligned}\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3)(\breve{\mathbb{P}}_2+\breve{\mathbb{P}}_3)\mathbb{Z}_1 &= \kappa_1\mathbb{D}_q\mathbb{X}_3\mathbb{F}\mathbb{X}\mathbb{F}\mathbb{Z}+\kappa_0\mathbb{D}_q\mathbb{X}_3\mathbb{F}(\sigma\mathbb{K}_1+\eta\mathbb{D})\mathbb{F}\mathbb{Z}\\ &= \kappa_1\mathbb{D}_q\mathbb{X}_3\mathbb{F}\mathbb{X}\mathbb{F}\mathbb{Z}+\kappa_1\mathbb{D}_q\mathbb{X}_2\mathbb{F}\mathbb{X}_3\mathbb{F}\mathbb{Z}+\eta\kappa_0\mathbb{D}_q\mathbb{X}_3\mathbb{F}\mathbb{Z}.\end{aligned}$$

Above we have used Lemma 4.2.2 applied to $\mathbb{X}_3 \in \mathcal{Q}_2$ and $\mathbb{F}\mathbb{K}_1 \in \mathcal{Q}_0$ and the fact that $\mathcal{S}^2(\mathbb{F}\mathbb{K}_1) = \mathbb{X}_2\mathbb{F}$. By the definition of $\mathbb{X}$, we get

$$\begin{aligned}\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3)(\breve{\mathbb{P}}_2+\breve{\mathbb{P}}_3)\mathbb{Z}_1 &= \kappa_1\mathbb{D}_q(\mathbb{X}\mathbb{F})^2\mathbb{Z}-\kappa_1\mathbb{D}_q(\mathbb{X}_2\mathbb{F})^2\mathbb{Z}+\eta\kappa_0\mathbb{D}_q\mathbb{X}\mathbb{F}\mathbb{Z}-\eta\kappa_0\mathbb{D}_q\mathbb{X}_2\mathbb{F}\mathbb{Z}\\ &= \kappa_0\mathbb{D}_q\mathcal{R}(\mathbb{X})\mathbb{Z}-\kappa_0\mathbb{D}_q\mathcal{R}(\mathbb{X}_2)\mathbb{Z},\end{aligned}$$

see (2.24). Adding $\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3)\breve{\mathbb{P}}_1\mathbb{Z}_1$ to both sides of the above equation and using the second identity from Lemma 5.2.3, we obtain

$$\mathcal{S}^2(\mathbb{T}_3)\mathcal{S}(\widetilde{\mathbb{X}}_3)\breve{\mathbb{P}}\mathbb{Z}_1 = \kappa_0\mathbb{D}_q\mathcal{R}(\mathbb{X})\mathbb{Z}+\kappa_0\mathcal{S}^2(\mathbb{T}_3)\mathbb{D}_q\mathcal{R}(\mathbb{X}_2)\mathbb{D}_q\mathbb{T}_3^{-1}\mathbb{Z}_1-\kappa_0\mathbb{D}_q\mathcal{R}(\mathbb{X}_2)\mathbb{Z}.$$

Lemma 4.2.2 (used with $\mathbb{D}_q\mathcal{R}(\mathbb{X}_2)\mathbb{D}_q \in \mathcal{Q}_2$ and $\mathbb{T}_3 \in \mathcal{Q}_0$), together with formulas (4.4) and (5.10), ends the proof. □

6.3. Assumption A3

We will first prove an auxiliary equality that, after some rearrangement, shows that Assumption A3 is implied by (1.9) with $\sigma\tau = 0$.

Lemma 6.3.1. *We have the following identity:*

$$\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{Z}_3\mathbb{P} = \mathbb{U}\mathbb{S}\mathbb{F}\mathcal{S}(\mathbb{D}_q)-qz\mathbb{Q}\mathbb{P}+\tfrac{1}{1+\sigma t}\mathbb{U}(\mathbb{E}-\mathbb{F}\mathbb{D})\mathbb{Z}_3\mathbb{P}.$$

Proof. Lemma 5.2.5 with (5.29) yields

$$\mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{X}}\mathbb{F}\mathbb{D}_q)\mathbb{P} = \mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{P}})\mathbb{X}\mathbb{D}_q\mathbb{F}. \tag{6.9}$$

According to Remark 4.2.9, we have

$$[\widetilde{\mathbb{P}}, \mathbb{F}\mathbb{D}_q] = \mathbb{Z}_2\widetilde{\mathbb{P}}\mathbb{F}\mathbb{D}_q - \mathbb{F}\mathbb{D}_q\widetilde{\mathbb{P}}\mathbb{Z}_2,$$

recall formula (5.5) for $k = 2$. Consequently, Lemma 5.2.4 gives

$$\mathcal{S}(\mathbb{F}\mathbb{D}_q\widetilde{\mathbb{P}}\mathbb{Z}_2 - \mathbb{Z}_2\widetilde{\mathbb{P}}\mathbb{F}\mathbb{D}_q + \mathbb{F}\mathbb{D}_q\mathbb{X}\mathbb{Z}_2 + z\mathbb{Q})\mathbb{P} = \mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{X}}\mathbb{F}\mathbb{D}_q)\mathbb{P} = \mathcal{S}(\mathbb{Z}_2\widetilde{\mathbb{P}})\mathbb{X}\mathbb{D}_q\mathbb{F},$$

where the last equality follows from (6.9). As a result, see (5.11), (5.31), and (6.1),

$$\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{Z}_3\mathbb{P} - \mathbb{U}\mathbb{D}_q\mathbb{F}\mathbb{P} + qz\mathbb{Q}\mathbb{P} = \mathbb{U}\mathbb{X}\mathbb{D}_q\mathbb{F}.$$

Taking into account formulas (5.29) and (6.1), we obtain

$$\begin{aligned}\mathbb{D}_q\mathbb{S}\mathbb{F}\mathbb{Z}_3\mathbb{P} + qz\mathbb{Q}\mathbb{P} = \mathbb{U}\mathbb{S}\mathbb{D}_q\mathbb{F} &= \mathbb{U}\mathbb{S}\mathbb{F}\mathcal{S}(\mathbb{D}_q) + \mathbb{U}\mathbb{S}(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{D}_q\mathbb{F}\\ &= \mathbb{U}\mathbb{S}\mathbb{F}\mathcal{S}(\mathbb{D}_q) + \mathbb{U}(\mathbb{E} - \mathbb{F}\mathbb{D}),\end{aligned} \tag{6.10}$$

where the last equality is satisfied because of (6.2) and (5.3). Moreover, formula (5.5) for $k = 3$ and identity (5.3) give

$$(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{Z}_3 = (\mathbb{E} - \mathbb{F}\mathbb{D})(\mathbb{E} + \sigma t\mathbb{D}_q\mathbb{F}) = (1 + \sigma t)(\mathbb{E} - \mathbb{F}\mathbb{D}),$$

and (5.30) implies that $(\mathbb{E} - \mathbb{F}\mathbb{D})\mathbb{Z}_3\mathbb{P} = (1 + \sigma t)(\mathbb{E} - \mathbb{F}\mathbb{D})$. Since $1 + \sigma t > 0$, we can divide by this factor and insert the last expression into (6.10). This gives the desired result. □

From (5.11) and (5.13), it follows that

$$\mathbb{D}_q + q\mathbb{Q}\mathbb{Z}_3^{-1} = \mathcal{S}(\mathbb{F}\mathbb{D}_q\mathbb{D}\mathbb{Z}_2 + \mathbb{Q})\mathbb{Z}_3^{-1} = \mathcal{S}(\mathbb{Z})\mathbb{Z}_3^{-1}.$$

Thus, the expression from Lemma 6.3.1, multiplied from the left by $\mathbb{U}^{-1}$ and from the right by $\mathbb{P}^{-1}\mathbb{Z}_3^{-1}$, yields, in view of Lemma 5.2.2, the following:

$$\mathbb{U}^{-1}\mathbb{D}_q\mathbb{S}\mathbb{F} - \mathbb{S}\mathbb{F}\mathbb{U}^{-1}\mathbb{D}_q = q(\mathbb{S}\mathbb{F} - z\mathbb{E})\mathbb{U}^{-1}\mathbb{Q}\mathbb{Z}_3^{-1} + \tfrac{1}{1+\sigma t}(\mathbb{E} - \mathbb{F}\mathbb{D}).$$

Hence we get the identity from Assumption A3 with $\widetilde{\mathbb{U}} = q\mathbb{U}^{-1}\mathbb{Q}\mathbb{Z}_3^{-1}$.

Chapter 7

Free quadratic harnesses

In this chapter, we will study free quadratic harnesses $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$. The adjective 'free' comes from the relations of this process with free probability, especially with free convolutions when $\sigma\tau = 0$, see [21, Section 4.3] for more details.

Our main aim is to describe the measure $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$ appearing in Theorem 1.6.1 in the case of the free quadratic harness. Our description gives a more explicit formula for the infinitesimal generator of $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$.

In the first section, we show that for all $x \in \mathbb{R}$, the polynomials $\{\widetilde{W}_n(\cdot;x,t)\}_{n=0}^{\infty}$ are orthogonal with respect to a probability measure. We provide an explicit form of this measure.

In the second part, we represent this measure as a (modified) univariate distribution of the considered free quadratic harness. This representation, which was previously obtained in Remark 4.2 of [24], is valid only for $t > 0$. In the concluding part of this chapter, we will provide another representation of $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$ that holds for all $t \geqslant 0$.

7.1. Description of the orthogonality measure $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$

If $q = -\sigma\tau$, then the assumptions (1.9) are reduced to the condition

$$0 \leqslant \sigma\tau < 1, \tag{7.1}$$

and therefore ξ from (1.21) is equal to $\xi = 2(1-\sigma\tau)$. The remaining parameters defined in (1.22) take the form

$$\xi_0 = \tfrac{(\tau+t)(1+\sigma t)}{(1-\sigma\tau)^2}, \qquad \xi_1 = \tfrac{\sigma(t+\tau)}{1-\sigma\tau}, \qquad \text{and} \qquad \xi_2 = \tfrac{\theta-\eta t}{1-\sigma\tau}.$$

Moreover, $\widetilde{q}$ given in (1.24) is zero, hence $[n]_{\widetilde{q}} = 1$ for all $n \in \mathbb{N}$. As a result,

$$\widetilde{\gamma}_n = \xi_2 + \tfrac{\eta\xi_0}{1+\xi_1} = \tfrac{\eta\tau+\theta}{1-\sigma\tau}, \qquad n \in \mathbb{N},$$

and

$$\sigma\widetilde{\gamma}_n + \eta = \tfrac{\eta+\sigma\theta}{1-\sigma\tau}, \qquad n \in \mathbb{N}.$$

Denote

$$\chi_1 := \tfrac{\eta+\theta\sigma}{1-\sigma\tau} \qquad \text{and} \qquad \chi_2 := \tfrac{\theta+\eta\tau}{1-\sigma\tau}, \tag{7.2}$$

and recall that $[0]_{\widetilde{q}} = 0$. Thus

$$\widetilde{a}_0(x) = \chi_2, \qquad \widetilde{a}_n(x) = \tfrac{\xi_0}{1+\xi_1}(\sigma\widetilde{\gamma}_{n+1} + \chi_1) + \chi_2, \qquad n \geqslant 1,$$

and

$$\widetilde{b}_1(x) = \tfrac{\xi_0}{1+\xi_1}(1 + \chi_1\chi_2), \qquad \widetilde{b}_n(x) = \xi_0(1 + \chi_1\chi_2), \qquad n \geqslant 2.$$

Therefore, $\{\widetilde{W}_n(y; x, t)\}_{n=0}^{\infty}$ satisfies the following three-step recurrence:

$$\begin{aligned}
\widetilde{W}_0(y; x, t) &= 1, \qquad \widetilde{W}_1(y; x, t) = y - \chi_2, \\
y\widetilde{W}_1(y; x, t) &= \widetilde{W}_2(y; x, t) + \widetilde{a}\widetilde{W}_1(y; x, t) + \tfrac{\widetilde{b}(1-\sigma\tau)}{1+\sigma t}\widetilde{W}_0(y; x, t), \\
y\widetilde{W}_n(y; x, t) &= \widetilde{W}_{n+1}(y; x, t) + \widetilde{a}\widetilde{W}_n(y; x, t) + \widetilde{b}\widetilde{W}_{n-1}(y; x, t), \qquad n \geqslant 2,
\end{aligned}$$

where $\widetilde{a} := \frac{(\sigma\chi_2+\chi_1)t+\tau\chi_1+\chi_2}{1-\sigma\tau}$ and $\widetilde{b} := \frac{(t+\tau)(1+\sigma t)}{(1-\sigma\tau)^2}(1 + \chi_1\chi_2)$.
Note that the coefficients in the three-step recurrence do not depend on x, so the polynomials do not depend on x as well. As a result, the same applies to the moment functional that makes these polynomials orthogonal.
In particular, we know from Theorem 1.6.1 that for x satisfying $1 + \eta x + \sigma x^2 > 0$, the moment functional is non-negative definite, so Remark A.0.2 implies that

$$\widetilde{b}_1(x) = \tfrac{t+\tau}{1-\sigma\tau}(1 + \chi_1\chi_2) \geqslant 0$$

for all $t \geqslant 0$. Since $\tau \geqslant 0$ (recall (1.5)) and (7.1) is satisfied, the above inequality implies that

$$1 + \chi_1\chi_2 \geqslant 0. \tag{7.3}$$

Under (7.3), the quadratic harness $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$ was constructed in [20, Theorem 1.1.]. Furthermore, from (7.3) we conclude that the free quadratic harness $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$ (with all moments finite) does not exist when $1+\chi_1\chi_2<0$.
When (7.3) holds, $\{\widetilde{W}_n(y;x,t)\}_{n=0}^{\infty}$ are orthogonal with respect to some probability measure $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$. Since orthogonal polynomials are determined up to multiplicative constants, polynomials $\{\widetilde{V}_n(y;x,t)\}_{n=0}^{\infty}$ satisfying the following three-step recurrence:

$$\begin{aligned}\widetilde{V}_0(y;x,t) &= \tfrac{1-\sigma\tau}{1+\sigma t}, \qquad \widetilde{V}_1(y;x,t) = y-\chi_2,\\ y\widetilde{V}_n(y;x,t) &= \widetilde{V}_{n+1}(y;x,t)+\widetilde{a}\widetilde{V}_n(y;x,t)+\widetilde{b}\widetilde{V}_{n-1}(y;x,t), \qquad n\geqslant 1,\end{aligned}$$

are orthogonal with respect to the same probability measure as the polynomials $\{\widetilde{W}_n(\cdot;x,t)\}_{n=0}^{\infty}$.
If $\widetilde{b}=0$, then $\{\widetilde{V}_n(\cdot;x,t)\}_{n=0}^{\infty}$ are orthogonal with respect to the Dirac measure concentrated at χ_2.
If $\widetilde{b}>0$, then the orthogonality measure for $\{\widetilde{V}_n(\cdot;x,t)\}_{n=0}^{\infty}$ is fully described in Theorem 3 in [28]. It has an absolutely continuous part and possibly (for some parameters of the quadratic harness) at most two atoms. Indeed, for the function f introduced in [28, Theorem 3], we have after simplification that

$$f(z) = \tfrac{t+\tau}{1+\sigma t}(\sigma z^2+\eta z+1).$$

Therefore f has two real roots $z^{\pm} = \frac{-\eta\pm\sqrt{\eta^2-4\sigma}}{2\sigma}$ if only $\eta^2>4\sigma>0$ (the superscript $+$ or $-$ indicates that we are taking $+$ or $-$, respectively, in each expression). If $\eta^2>4\sigma=0$, then f has a root in $-\frac{1}{\eta}$. As a result, with $(z)_+ := \frac{z+|z|}{2}$ for $z\in\mathbb{R}$, the measure $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$ can be decomposed as follows:

i) the absolutely continuous part μ_c is proportional to

$$\frac{\sqrt{4\widetilde{b}-(y-\widetilde{a})^2}}{\sigma y^2+\eta y+1}\mathbb{1}_{\left(\widetilde{a}-2\sqrt{\widetilde{b}},\widetilde{a}+2\sqrt{\widetilde{b}}\right)}(y)\mathrm{d}y,$$

ii) the discrete part μ_d may appear only in two cases:

a) if $\eta^2>4\sigma>0$, then a discrete part is proportional to

$$\left(\tfrac{1+\chi_1\chi_2}{|z^+-\chi_2|}-\tfrac{(1+\sigma t)|z^+-\chi_2|}{t+\tau}\right)_+\delta_{z^+}(\mathrm{d}y)+\left(\tfrac{1+\chi_1\chi_2}{|z^--\chi_2|}-\tfrac{(1+\sigma t)|z^--\chi_2|}{t+\tau}\right)_+\delta_{z^-}(\mathrm{d}y),$$

b) if $\eta^2 > 4\sigma = 0$, then a discrete part is proportional to

$$\left(1 - \tfrac{\widetilde{b}}{\eta^2(t+\tau)^2}\right)_+ \delta_{-\frac{1}{\eta}}(\mathrm{d}y),$$

iii) there is no singular part μ_s.

Therefore, the orthogonality measure can be written as:

$$\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}(\mathrm{d}y) = \zeta_c\mu_c(\mathrm{d}y) + \zeta_d\mu_d(\mathrm{d}y),$$

where ζ_c and ζ_d are some real constants uniquely determined by the requirement that $\{\widetilde{V}_n(\cdot;x,t)\}_{n=0}^{\infty}$ are orthogonal with respect to $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$. In particular, integrating $\widetilde{V}_0(\cdot;x,t)$ and $\widetilde{V}_1(\cdot;x,t)$ with respect to $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$ gives $\frac{1-\sigma\tau}{1+\sigma t}$ and 0, respectively.
Note that if $\sigma = \eta = 0$, then

$$\nu_{x,t,0,\theta,0,\tau,0}(\mathrm{d}y) = \tfrac{1}{2\pi(t+\tau)}\sqrt{4(t+\tau) - (y-\theta)^2}\,\mathbb{1}_{\left(\theta-2\sqrt{t+\tau},\theta+2\sqrt{t+\tau}\right)}(y)\mathrm{d}y,$$

that is $\nu_{x,t,0,\theta,0,\tau,0}$ is a probability density function of a Wigner semicircle distribution with mean θ and variance $t+\tau$.

7.2. Relation to the univariate distributions

We continue under assumptions (7.1) and (7.3). Recall that for free quadratic harnesses, the polynomials $\{\widetilde{W}_n(y;x,t)\}_{n=0}^{\infty}$ do not depend on x. Therefore, the probabilistic orthogonality measure $\nu_{x,t,\sigma,\tau,\eta,\theta,-\sigma\tau}$ also does not depend on x.

Now we will represent $\{\widetilde{W}_n(y;x,t)\}_{n=0}^{\infty}$ in terms of the monic martingale polynomials $\{p_n(y;t)\}_{n=0}^{\infty}$ for $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$, which satisfy the following three-step recurrence (see Proposition 2.2 in [20]):

$$\begin{aligned}
p_0(y;t) &= 1, \qquad p_1(y;t) = y,\\
yp_1(y;t) &= p_2(y;t) + (\chi_1 t + \chi_2)p_1(y;t) + tp_0(y;t),\\
yp_2(y;t) &= p_3(y;t) + \widetilde{a}p_2(y;t) + \widetilde{b}(1-\sigma\tau)p_1(y;t),\\
yp_n(y;t) &= p_{n+1}(y;t) + \widetilde{a}p_n(y;t) + \widetilde{b}p_{n-1}(y;t), \qquad n \geqslant 3.
\end{aligned} \tag{7.4}$$

We will also consider polynomials $\{U_n(y;t)\}_{n=0}^{\infty}$ defined by

$$U_n(y,t) := \begin{cases} \tau p_2(y;t) + \chi_2(t+\tau)p_1(y;t) + t(t+\tau)p_0(y;t), & n=0,\\ \tau p_3(y;t) + \frac{(\tau+t)(\chi_1\tau+\chi_2)}{1-\sigma\tau}p_2(y;t) + \frac{(t+\tau)(1-\sigma\tau)\widetilde{b}}{1+\sigma t}p_1(y;t), & n=1,\\ \tau p_{n+2}(y;t) + \frac{(\tau+t)(\chi_1\tau+\chi_2)}{1-\sigma\tau}p_{n+1}(y;t) + \frac{(t+\tau)\widetilde{b}}{1+\sigma t}p_n(y;t), & n\geqslant 2. \end{cases}$$

It turns out that U_n is connected with $\widetilde{W}_n$ in the following way:

Lemma 7.2.1. *For all $n \in \mathbb{N}_0$ we have*

$$U_n(y;t) = (t^2+\theta ty+\tau y^2)\widetilde{W}_n(y;x,t).$$

Proof. A direct calculation shows that the assertion is true for $n=0,1$, i.e., it is easy to check that

$$U_0(y;t) = t^2+\theta ty+\tau y^2 \qquad \text{and} \qquad U_1(y;t) = (t^2+\theta ty+\tau y^2)(y-\chi_2).$$

Next, the definition of U_1 and the three-step recurrence for $\{p_n(y;t)\}_{n=0}^{\infty}$ imply

$$\begin{aligned} yU_1(y;t) =& \tau(p_4(y;t)+\widetilde{a}p_3(y;t)+\widetilde{b}p_2(y;t))\\ &+\tfrac{(\tau+t)(\chi_1\tau+\chi_2)}{1-\sigma\tau}(p_3(y;t)+\widetilde{a}p_2(y;t)+\widetilde{b}(1-\sigma\tau)p_1(y;t))\\ &+\tfrac{(t+\tau)(1-\sigma\tau)}{1+\sigma t}\widetilde{b}yp_1(y;t). \end{aligned}$$

Using the definition of U_n for $n=0,1,2$ and collecting the expressions with p_2, p_1, and p_0, respectively, we get

$$\begin{aligned} yU_1(y;t) =& U_2(y;t)+\widetilde{a}U_1(y;t)+\tfrac{\widetilde{b}(1-\sigma\tau)}{1+\sigma t}U_0(y;t) - \left(\tfrac{1-\sigma\tau}{1+\sigma t}\tau+\tfrac{t+\tau}{1+\sigma t}-\tau\right)\widetilde{b}p_2(y;t)\\ &-\left(\tfrac{1-\sigma\tau}{1+\sigma t}\chi_2+\widetilde{a}\tfrac{1-\sigma\tau}{1+\sigma t}-(\chi_1\tau+\chi_2)\right)(t+\tau)\widetilde{b}p_1(y;t)\\ &-\tfrac{1-\sigma\tau}{1+\sigma t}(t+\tau)t\widetilde{b}p_0(y;t)+\tfrac{(t+\tau)(1-\sigma\tau)}{1+\sigma t}\widetilde{b}yp_1(y;t). \end{aligned}$$

After simplification and in view of the second line in (7.4) we get

$$yU_1(y;t) = U_2(y;t)+\widetilde{a}U_1(y;t)+\tfrac{\widetilde{b}(1-\sigma\tau)}{1+\sigma t}U_0(y;t).$$

Analogously, we can obtain

$$yU_2(y;t) = U_3(y;t) - \tfrac{(t+\tau)t}{1+\sigma t}p_3(y;t) + \widetilde{a}\left(U_2(y;t) - \tfrac{\widetilde{b}(t+\tau)}{1+\sigma t}p_2(y;t)\right) + \widetilde{b}\left(U_1(y;t) - \tfrac{\widetilde{b}(t+\tau)(1-\sigma\tau)}{1+\sigma t}p_1(y;t)\right) + \tfrac{(t+\tau)\widetilde{b}}{1+\sigma t}yp_2(y;t).$$

The third formula in (7.4) implies $yU_2(y;t) = U_3(y;t) + \widetilde{a}U_2(y;t) + \widetilde{b}U_1(y;t)$. Moreover, directly from the last line in (7.4) we see that

$$yU_n(y;t) = U_{n+1}(y;t) + \widetilde{a}U_n(y;t) + \widetilde{b}U_{n-1}(y;t), \qquad n \geqslant 3.$$

Since the sequence $\{U_n(y;t)\}_{n=0}^{\infty}$ satisfies the same three-step recurrence as $\{(t^2 + \theta ty + \tau y^2)\widetilde{W}_n(y;x,t)\}_{n=0}^{\infty}$ with the same initial conditions, we get the desired result. □

Let π_t be a univariate distribution of $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$ at time $t>0$. The martingale polynomials $\{p_n(\cdot;t)\}_{n=0}^{\infty}$ are orthogonal with respect to π_t. Consequently, Lemma 7.2.1 implies

$$\int_{\mathbb{R}} \tfrac{t^2+\theta ty+\tau y^2}{t(t+\tau)}\widetilde{W}_n(y;x,t)\pi_t(\mathrm{d}y) = \mathbb{1}_{\{n=0\}},$$

where above we have used the fact that $\int_{\mathbb{R}} p_n(y;t)\pi_t(\mathrm{d}y) = \mathbb{1}_{\{n=0\}}$. Then Exercise 4.14 in [27] yields:

Corollary 7.2.2. *Polynomials $\{\widetilde{W}_n(\cdot;x,t)\}_{n=0}^{\infty}$ associated with the infinitesimal generator of $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$ are orthogonal with respect to $\frac{t^2+\theta ty+\tau y^2}{t(t+\tau)}\pi_t(\mathrm{d}y)$, $t>0$, where π_t is the univariate distribution of $QH(\eta,\theta;\sigma,\tau;-\sigma\tau)$. That is,*

$$\nu_{x,t,\sigma,\tau,\eta,\theta,-\sigma\tau}(\mathrm{d}y) = \tfrac{t^2+\theta ty+\tau y^2}{t(t+\tau)}\pi_t(\mathrm{d}y). \tag{7.5}$$

Consequently, in the case of free quadratic harnesses, Theorem 1.6.1 coincides with [24, Remark 4.2], which was established using an alternative method based on the Cauchy-Stieltjes transformation.

However, the representation (7.5) does not work for $t=0$, since then $\pi_t(\mathrm{d}y) = \delta_0(\mathrm{d}y)$, see (1.2), and

$$\int_{\mathbb{R}}(t^2+\theta ty+\tau y^2)\widetilde{W}_n(y;x,t)\pi_t(\mathrm{d}y) = \int_{\mathbb{R}}\tau y^2\widetilde{W}_n(y;x,0)\delta_0(\mathrm{d}y) = 0, \qquad n\in\mathbb{N}_0,$$

which makes the normalization impossible for $n = 0$.

To overcome this normalization problem and determine the measure $\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}$ for all $t \geqslant 0$, we will consider a free quadratic harness with slightly modified parameters. Namely, we obtain the following result:

Proposition 7.2.3. *Let $\widetilde{\pi}_t$ be an univariate distribution of $QH(\eta, \chi_2; \sigma, 0; 0)$ at time $t \geqslant 0$ (recall (7.2)). Then the infinitesimal generator of $QH(\eta, \theta; \sigma, \tau; -\sigma\tau)$ acting on polynomials can be represented as in Theorem 1.6.1 with*

$$\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}(\mathrm{d}y) = \left(1 + \tfrac{\theta+\eta\tau}{t+\tau} y\right) \widetilde{\pi}_{\frac{t+\tau}{1-\sigma\tau}}(\mathrm{d}y), \qquad \text{when } t + \tau > 0,$$

and

$$\nu_{x,t,\eta,\theta,\sigma,\tau,-\sigma\tau}(\mathrm{d}y) = \delta_{\chi_2}(\mathrm{d}y), \qquad \text{when } t = \tau = 0.$$

Proof. Denote by $\{V_n(y; x, t)\}_{n=0}^{\infty}$ the polynomials associated with infinitesimal generator of $QH(\eta, \chi_2; \sigma, 0; 0)$, i.e., $\{V_n(y; x, t)\}_{n=0}^{\infty}$ satisfies the following three-step recurrence:

$$\begin{aligned}
V_0(y; x, t) &= 1, \qquad V_1(y; x, t) = y - \chi_2,\\
yV_1(y; x, t) &= V_2(y; x, t) + \breve{a}V_1(y; x, t) + \tfrac{\breve{b}}{1+\sigma t} V_0(y; x, t),\\
yV_n(y; x, t) &= V_{n+1}(y; x, t) + \breve{a}V_n(y; x, t) + \breve{b}V_{n-1}(y; x, t), \qquad n \geqslant 2,
\end{aligned}$$

where

$$\begin{aligned}
\breve{a} &= (\sigma\chi_2 + \eta + \sigma\chi_2)t + \chi_2 = (\sigma\chi_2 + \chi_1)t + \chi_2,\\
\breve{b} &= t(1 + \sigma t)(1 + (\eta + \sigma\chi_2)\chi_2) = t(1 + \sigma t)(1 + \chi_1\chi_2).
\end{aligned}$$

Comparing this with the three-step recurrences for $\{\widetilde{W}_n(y; x, t)\}_{n=0}^{\infty}$ leads to the conclusion that

$$\widetilde{W}_n(y; x, t) = V_n(y; x, \widetilde{t}), \qquad n \in \mathbb{N}_0,$$

where $\widetilde{t} := \frac{t+\tau}{1-\sigma\tau}$.

As we have already proved, the polynomials $\{V_n(\cdot; x, \widetilde{t})\}_{n=0}^{\infty}$ are orthogonal with respect to $\left(1 + y\chi_2/\widetilde{t}\right) \widetilde{\pi}_{\widetilde{t}}(\mathrm{d}y)$ if only $\widetilde{t} > 0$ (equivalently $t + \tau > 0$), see Corollary 7.2.2.

If $t = \tau = 0$, then the three-step recurrence for $\{\widetilde{W}_n(\cdot; x, 0)\}_{n=0}^{\infty}$ significantly simplifies, in particular $\widetilde{b}_0 = 0$. Hence according to Theorem A.1 in [22], $\{\widetilde{W}_n(\cdot; x, 0)\}_{n=0}^{\infty}$ are orthogonal with respect to the Dirac measure concentrated at χ_2. $\square$

It is worth emphasizing that Proposition 7.2.3 is true for $0 \leqslant \sigma\tau < 1$. Observe that for $\sigma\tau > 0$ its assertion is covered by Proposition 2.2.1. However, Proposition 2.2.1 does not cover the case $\sigma\tau = 0$. In particular, the result obtained in Proposition 7.2.3 is not evident in the case of $\sigma = 0$ and $\tau > 0$. For this reason, we decided to include the proof of Proposition 7.2.3, even though it partially covers the previously considered case of Proposition 2.2.1.

Chapter 8

Quadratic harnesses with $q = -1$

In this chapter, we will examine a quadratic harness with $q = -1$. We will show that for each $t > 0$, X_t can only take on two distinct values. Our main goal is to construct this process in previously unknown cases and to derive a formula for the infinitesimal generator directly from the definition. Additionally, we will compare the derived formula with the one implied by Theorem 3.3.1.

The construction of quadratic harnesses with $q = -1$ was previously carried out in [19, Section 3.2], only in the case $\sigma = \tau = 0$ (the bi-Poisson process case).

Before we tackle the general case of $q = -1$ (which will be done in Section 8.2), we will devote the next section to analyzing the bi-Poisson process with $q = -1$.

8.1. Bi-Poisson process $QH(\eta, \theta; 0, 0; -1)$

Let us now recall the construction from [19, Section 3.2] in detail. Assume that

$$1 + \eta\theta \geqslant 0.$$

and consider a Markov process $(X_t)_{t\geqslant 0}$, starting from zero, with univariate distributions given by

$$\mathbb{P}(X_t = x_{t,+}) = \tfrac{p_-(t)}{2y(t)}, \qquad \mathbb{P}(X_t = x_{t,-}) = \tfrac{p_+(t)}{2y(t)},$$

where $x_{t,\pm} := \frac{1}{2}(\theta + \eta t \pm y(t))$ and

$$p_\pm(t) := y(t) \pm (\theta + \eta t).$$

with $y(t) := \sqrt{4t + (\theta + \eta t)^2}$. The form of the support ensures that for all $t \geqslant 0$

$$X_t^2 - (\theta + \eta t)X_t = t \qquad \text{a.s.} \tag{8.1}$$

The transition probabilities for $0 < s < t$ are given by

$$\begin{aligned} \mathbb{P}(X_t = x_{t,+}|X_s = x_{s,+}) &= \tfrac{p_-(t)+p_+(s)}{2y(t)}, \quad \mathbb{P}(X_t = x_{t,-}|X_s = x_{s,+}) = \tfrac{p_+(t)-p_+(s)}{2y(t)}, \\ \mathbb{P}(X_t = x_{t,+}|X_s = x_{s,-}) &= \tfrac{p_-(t)-p_-(s)}{2y(t)}, \quad \mathbb{P}(X_t = x_{t,-}|X_s = x_{s,-}) = \tfrac{p_+(t)+p_-(s)}{2y(t)}. \end{aligned} \tag{8.2}$$

Such a process is a quadratic harness $QH(\eta, \theta; 0, 0; -1)$, meaning that $(X_t)_{t \geqslant 0}$ satisfies (1.2) and (1.1), and for all $0 \leqslant s < t < u$,

$$\begin{aligned} \mathbb{E}(X_t^2|\mathcal{F}_{s,u}) =& \mathbb{E}(X_t^2|X_s, X_u) \\ =& \tfrac{(u-t)(u+t)}{(u-s)(u+s)}X_s^2 + \tfrac{(t-s)(t+s)}{(u-s)(u+s)}X_u^2 \\ &+ \tfrac{(u-t)(t-s)}{(u-s)(u+s)}(\eta u - \theta)X_s + \tfrac{(u-t)(t-s)}{(u-s)(u+s)}(\theta - \eta s)X_u + \tfrac{(u-t)(t-s)}{u+s}. \end{aligned} \tag{8.3}$$

8.1.1. Infinitesimal generator by direct calculation

We will now obtain a formula for the infinitesimal generator of $QH(\eta, \theta; 0, 0; -1)$ in a direct way (without appealing to Theorem 3.3.1) using the above explicit construction. For $t > 0$

$$\begin{aligned} \mathbb{E}(f(X_{t+h})|X_t = x_{t,+}) =& \tfrac{f(x_{t+h,+})(p_-(t+h)+p_+(t)) + f(x_{t+h,-})(p_+(t+h)-p_+(t))}{2y(t+h)} \\ =& \tfrac{f(x_{t+h,-})}{2y(t+h)}(y(t+h) - y(t)) - \tfrac{f(x_{t+h,+})}{2y(t+h)}(y(t+h) - y(t)) \\ &+ \eta h \tfrac{f(x_{t+h,-})}{2y(t+h)} - \eta h \tfrac{f(x_{t+h,+})}{2y(t+h)} + f(x_{t+h,+}). \end{aligned}$$

If f is differentiable at $x_{t,+}$ and continuous at $x_{t,-}$, then the limit on the right-hand side of (1.15) exists and

$$\begin{aligned} (\mathbf{A}_t^+ f)(x_{t,+}) =& \tfrac{f(x_{t,-})}{2y(t)}y'(t) - \tfrac{f(x_{t,+})}{2y(t)}y'(t) + \tfrac{\eta}{2y(t)}(f(x_{t,-}) - f(x_{t,+})) + f'(x_{t,+})\tfrac{\mathrm{d}}{\mathrm{d}t}(x_{t,+}) \\ =& \left(\tfrac{1}{2}\eta + \tfrac{2+\eta(\theta+\eta t)}{2y(t)}\right) \tfrac{f(x_{t,-})-f(x_{t,+})}{y(t)} + f'(x_{t,+}) \left(\tfrac{1}{2}\eta + \tfrac{2+\eta(\theta+\eta t)}{2y(t)}\right). \end{aligned}$$

Since $\frac{1}{2}\eta + \frac{2+\eta(\theta+\eta t)}{2y(t)} = \frac{1+\eta x_{t,+}}{y(t)}$, we finally obtain

$$(\mathbf{A}_t^+ f)(x_{t,+}) = \frac{1+\eta x_{t,+}}{\sqrt{4t+(\theta+\eta t)^2}}\left(\frac{f(x_{t,-})-f(x_{t,+})}{\sqrt{4t+(\theta+\eta t)^2}} + f'(x_{t,+})\right).$$

Analogously, we can show that if f is differentiable at $x_{t,-}$ and continuous at $x_{t,+}$, then

$$(\mathbf{A}_t^+ f)(x_{t,-}) = \frac{1+\eta x_{t,-}}{\sqrt{4t+(\theta+\eta t)^2}}\left(\frac{f(x_{t,+})-f(x_{t,-})}{\sqrt{4t+(\theta+\eta t)^2}} - f'(x_{t,-})\right).$$

As a result, if f is differentiable at $x_{t,+}$ and $x_{t,-}$, then $f \in \mathcal{D}(\mathbf{A}_t^+)$.
Furthermore, for a fixed $t > 0$, we observe that $x_{t,\pm} \notin \operatorname{supp}(X_{t-h})$ for all $0 < h \leqslant t$. Therefore, for $0 < h \leqslant t$, we can choose

$$\mathbb{E}(X_t|X_{t-h} = x_{t,\pm}) := \mathbb{E}(X_{t+h}|X_t = x_{t,\pm}).$$

Consequently, if f is differentiable at $x_{t,+}$ and $x_{t,-}$, then $f \in \mathcal{D}(\mathbf{A}_t^-)$ and

$$(\mathbf{A}_t^- f)(x_{t,\pm}) = (\mathbf{A}_t^+ f)(x_{t,\pm}).$$

When $t = 0$, the expression under the limit on the right-hand side of (1.15) is equal to

$$\frac{\mathbb{E}(f(X_h)|X_0=0)-f(0)}{h} = \frac{(f(x_{h,+})-f(0))p_-(h)}{2hy(h)} + \frac{(f(x_{h,-})-f(0))p_+(h)}{2hy(h)}. \tag{8.4}$$

To find the limit of (8.4), we will consider several cases:

1) $\theta = 0$. If f is differentiable twice at 0, then by Taylor's theorem

$$\frac{\mathbb{E}(f(X_h)|X_0=0)-f(0)}{h} = \frac{\left(f'(0)x_{h,+}+\frac{1}{2}f''(0)x_{h,+}^2+o(x_{h,+}^2)\right)p_-(h)}{2hy(h)} + \frac{\left(f'(0)x_{h,-}+\frac{1}{2}f''(0)x_{h,-}^2+o(x_{h,-}^2)\right)p_+(h)}{2hy(h)},$$

where $o(x_{h,+}^2)$ is a function that divided by $x_{h,+}^2$ goes to zero when h goes to zero (interpretation of $o(x_{h,-}^2)$ is analogous). Furthermore, since $x_{h,+}p_-(h)+x_{h,-}p_+(h) = 0$,

$$\frac{\mathbb{E}(f(X_h)|X_0=0)-f(0)}{h} = \left(\frac{1}{2}f''(0) + \frac{o(x_{h,+}^2)}{x_{h,+}^2}\right)\frac{x_{h,+}^2 p_-(h)}{2hy(h)} + \left(\frac{1}{2}f''(0) + \frac{o(x_{h,-}^2)}{x_{h,-}^2}\right)\frac{x_{h,-}^2 p_+(h)}{2hy(h)}.$$

Since

$$\frac{x_{h,\pm}^2 p_\mp(h)}{2hy(h)} = \frac{1}{2}\left(1 \pm \frac{\eta h}{y(h)}\right),$$

the limit of the above expression exists as h goes to zero and therefore $f \in \mathcal{D}(\mathbf{A}_0^+)$ and

$$(\mathbf{A}_0^+ f)(0) = \tfrac{1}{2} f''(0).$$

Note that the existence of the first derivative at zero is not sufficient to ensure that f is in $\mathcal{D}(\mathbf{A}_0^+)$. For example, for $f(x) := |x|^{3/2}$, the expression in (8.4) goes to infinity when h goes to zero.

2) $\theta > 0$. Under this assumption, as h tends to zero, $x_{h,+}$ goes to θ while $x_{h,-}$ goes to 0. Since

$$\tfrac{p_-(h)}{2hy(h)} = \tfrac{2}{y(h)(\theta+\eta h+y(h))} \xrightarrow{h \to 0^+} \tfrac{1}{\theta^2},$$

and

$$\tfrac{x_{h,-}p_+(h)}{2hy(h)} = -\tfrac{1}{y(h)} \xrightarrow{h \to 0^+} -\tfrac{1}{\theta},$$

the limit of formula (8.4) when h goes to zero exists if only f is differentiable at 0 and continuous at θ. In such case, $f \in \mathcal{D}(\mathbf{A}_0^+)$

$$(\mathbf{A}_0^+ f)(0) = \tfrac{f(\theta)-f(0)}{\theta^2} - \tfrac{1}{\theta} f'(0).$$

3) $\theta < 0$. We proceed similarly as when $\theta > 0$ to obtain

$$(\mathbf{A}_0^+ f)(0) = \tfrac{f(\theta)-f(0)}{\theta^2} - \tfrac{1}{\theta} f'(0),$$

where f is differentiable at 0 and continuous at θ.

8.1.2. Infinitesimal generator by the algebraic approach

In this section, we clarify certain issues which arise with the approach used in [24] for the infinitesimal generator of the bi-Poisson process when $q = -1$. All the results from [24] were obtained under the assumption that the considered process is polynomial, with an infinite state space. This applies, in particular, to the q-commutation equation. We have found a solution to this equation in Theorem 1.6.1, also for $q = -1$. However, in order to apply Theorem 1.6.1 to the bi-Poisson process with $q = -1$, one needs to exercise more caution, as it is a process with a finite state space.

More specifically, for $QH(\eta, \theta; 0, 0; -1)$, the element $\mathbb{P}_{s,t} \in \mathcal{Q}$ given in [24, Definition 1.3], whose nth coordinate corresponds to $\mathbb{E}(X_t^n|X_s = x)$, $n \in \mathbb{N}_0$, is not uniquely determined (as described in [24, Section 1.1]). To illustrate this difficulty, let us consider the following example:

Example 8.1.1. *Let $(X_t)_{t\geqslant 0}$ be a bi-Poisson process with $q = -1$. Using* (8.2)*, tedious calculations show*

$$\mathbb{E}(X_t^3|X_s) = X_s^3 + \eta(t-s)X_s^2 + (t-s)[\eta(\eta t+\theta)+1]X_s + (t-s)(\eta t+\theta).$$

However, in view of (8.1) *we also have*

$$\mathbb{E}(X_t^3|X_s) = X_s^3 - (\theta+\eta s)X_s^2 + \left(t-s+(\theta+\eta t)^2\right)X_s + t(\theta+\eta t).$$

Hence, the third coordinate of $\mathbb{P}_{s,t}$ can be represented as the polynomial (in x)

$$x^3 + \eta(t-s)x^2 + (t-s)[\eta(\eta t+\theta)+1]x + (t-s)(\eta t+\theta)$$

and also as

$$x^3 - (\theta+\eta s)x^2 + \left(t-s+(\theta+\eta t)^2\right)x + t(\theta+\eta t). \tag{8.5}$$

However, if we impose the additional condition that the nth coordinate of $\mathbb{P}_{s,t}$ is a monic polynomial of degree n, then the first three polynomials of $\mathbb{P}_{s,t}$ are uniquely determined. They are then given by:

$$1, \qquad x, \qquad x^2 + \eta(t-s)x + t - s. \tag{8.6}$$

The uniqueness arises from the fact that for any two given points only one parabola of the form $x^2 + bx + c$ passes through them.

The problem with uniqueness implies that the family $\{\mathbb{P}_{s,t}\}_{0\leqslant s<t}$ may not satisfy the condition stated in [24, Definition 1.3], which serves as a foundation for the subsequent arguments in the cited paper.

Example 8.1.2 (Continuation of Example 8.1.1)**.** *Suppose that the first four polynomials for $\mathbb{P}_{s,t}$ are given by* (8.6) *and* (8.5)*. Then the third coordinate of $\mathbb{P}_{s,t}\mathbb{P}_{t,u}$ is equal to (recall*

(2.1)*):*

$$x^3 - (\theta + \eta s)x^2 + \left(t - s + (\theta + \eta t)^2\right) x + t(\theta + \eta t)$$
$$- (\theta + \eta t)(x^2 + \eta(t - s)x + t - s) + \left(u - t + (\theta + \eta u)^2\right) x + u(\theta + \eta u).$$

Upon simplification, we observe that the coefficient of x^2 is equal to $-(s + t)\eta - 2\theta$. By appropriately choosing values for s and t, we can ensure that this coefficient is not equal to $\theta + \eta s$. Consequently, the third coordinate of $\mathbb{P}_{s,t}\mathbb{P}_{t,u}$ does not coincide with the third coordinate of $\mathbb{P}_{s,u}$ for all $0 \leqslant s < t < u$.

Despite the aforementioned problems, we will now find a specific family $\{\mathbb{P}_{s,t}\}_{0\leqslant s<t}$ that satisfies the conditions stated in Definition 1.3 of [24], even in the case $q = -1$.
In [19], the authors introduced orthogonal polynomials $\{Q_n(y; x, t, s)\}_{n=0}^{\infty}$, $t > s \geqslant 0$, that are orthogonal with respect to the transition probabilities of the bi-Poisson process when $q \in (-1, 1)$. Since this process has an infinite state space, the corresponding elements $\mathbb{P}_{s,t}$ are uniquely determined. Specifically, the nth coordinate of $\mathbb{P}_{s,t}$, which represents $\mathbb{E}(X_t^n|X_s = x)$, can be expressed as a monic polynomial in the variable x in a unique manner.
The conditional expectations $\mathbb{E}(X_t^n|X_s = x)$ can be obtained recursively from the monic polynomials $\{Q_n(y; x, t, s)\}_{n=0}^{\infty}$ due to:

$$\mathbb{E}\left(Q_n(X_t; x, t, s)|X_s = x\right) = 0, \qquad n \in \mathbb{N}_0,$$

which follows from the fact that $\{Q_n(y; x, t, s)\}_{n=0}^{\infty}$ are orthogonal with respect to the conditional distributions.
The polynomials $\{Q_n(y; x, t, s)\}_{n=0}^{\infty}$ do depend on the parameter q. Moreover, the coefficients appearing in the three-step recurrence for $\{Q_n(y; x, t, s)\}_{n=0}^{\infty}$ have a limit as q approaches -1, see [19, (7) and (8)]. This allows us to define a new family of polynomials $\{\widetilde{Q}_n(y; x, t, s)\}_{n=0}^{\infty}$ such that

$$\widetilde{Q}_n(y; x, t, s) := \lim_{q\to-1^+} Q_n(y; x, t, s).$$

It turns out that the polynomials $\{\widetilde{Q}_n(y; x, t, s)\}_{n=0}^{\infty}$ satisfy the following three-step recurrence:

$$\begin{aligned}
\widetilde{Q}_{-1}(y; x, t, s) =& 0, \qquad \widetilde{Q}_0(y; x, t, s) = 1, \\
y\widetilde{Q}_n(y; x, t, s) =& \widetilde{Q}_{n+1}(y; x, t, s) + \left((-1)^n x + \tfrac{1-(-1)^n}{2}(\theta + \eta t)\right) \widetilde{Q}_n(y; x, t, s) \\
& + \tfrac{1-(-1)^n}{2}(t - s)(1 + \eta x)\widetilde{Q}_{n-1}(y; x, t, s), \qquad n \in \mathbb{N}_0.
\end{aligned}$$

It was proved in [19, Section 3.2] that $\{\widetilde{Q}_n(y; x, t, s)\}_{n=0}^{\infty}$ are orthogonal with respect to the transition probabilities of $QH(\eta, \theta; 0, 0; -1)$, see (8.2). Furthermore, we can construct the corresponding $\widetilde{\mathbb{P}}_{s,t}$ from $\{\widetilde{Q}_n(y; x, t, s)\}_{n=0}^{\infty}$ in the same way as we obtained $\mathbb{P}_{s,t}$ from $\{Q_n(y; x, t, s)\}_{n=0}^{\infty}$.

It is important to note that $\widetilde{\mathbb{P}}_{s,t}$ obtained this way coincides in fact with the limit of $\mathbb{P}_{s,t}$ as q approaches -1. This follows from the fact that the coordinates of $\mathbb{P}_{s,t}$ are obtained from the polynomials $\{Q_n(y; x, t, s)\}_{n=0}^{\infty}$, and all coefficients in each polynomial $Q_n(y; x, t, s)$ depend continuously on q.

Since the family $\{\mathbb{P}_{s,t}\}_{0\leqslant s<t}$ satisfies the conditions from the Definition 1.3 in [24], and the multiplication given in (2.1) is a continuous operation, $\{\widetilde{\mathbb{P}}_{s,t}\}_{0\leqslant s<t}$ also satisfies the same conditions.

Consequently, all arguments presented in [24] remain valid without any modification for $q = -1$. In particular, the q-commutation equation holds in this case. Therefore, we can apply the method described in Chapter 2 to derive a formula for the infinitesimal generator when $q = -1$. Thus, according to Theorems 1.6.1 and 3.3.1, for any function $g \in \mathcal{C}^2(\mathbb{R})$ (see the comment below Theorem 3.3.1), we have the following result:

$$(\mathbf{A}_t g)(x) = \begin{cases} \frac{1+\eta x}{2} g''(x) & \text{when } \theta + \eta t = 2x, \\ \frac{1+\eta x}{\theta+\eta t-2x}\left(\frac{g(\theta+\eta t-x)-g(x)}{\theta+\eta t-2x} - g'(x)\right) & \text{when } \theta + \eta t \neq 2x. \end{cases} \tag{8.7}$$

Indeed, the three-step recurrence for polynomials $\{\widetilde{W}_n(y; x, t)\}_{n=0}^{\infty}$ takes the form:

$$\begin{aligned}
&\widetilde{W}_0(y; x, t) = 1, \\
&\widetilde{W}_{n+1}(y; x, t) = \left(y + (-1)^n x - (\theta + \eta t)\tfrac{1-(-1)^n}{2}\right) \widetilde{W}_n(y; x, t), \quad n \geqslant 0.
\end{aligned}$$

Hence, according to [22, Theorem A.1], the polynomials $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$ are orthogonal

with respect to the Dirac measure $\delta_{\theta+\eta t-x}$ concentrated at $\theta+\eta t-x$ since $\widetilde{W}_1(y;x,t)=y+x-\theta-\eta t$.
It is easy to verify that the formula for the infinitesimal generator given in (8.7) coincides with the formulas derived in the previous subsection. However, it is worth noting that a range of admissible functions obtained through the algebraic approach is a proper subset of that obtained through direct calculations.

8.2. Quadratic harnesses $QH(\eta,\theta;\sigma,\tau;-1)$

Recall that we proceed under assumptions (1.9), which reduce to

$$0\leqslant\sigma\tau<1 \tag{8.8}$$

when $q=-1$. Recall the notation introduced in (7.2) and assume that

$$1+\chi_1\chi_2\geqslant 0. \tag{8.9}$$

For such χ_1 and χ_2, there exists a quadratic harness $QH(\chi_1,\chi_2;0,0;-1)$ denoted by $(Y_t)_{t\geqslant 0}$. It is noteworthy that $(Y_t)_{t\geqslant 0}$ is also a quadratic harness with modified parameters:

Lemma 8.2.1. *The stochastic process $(Y_t)_{t\geqslant 0}$ is $QH(\eta,\theta;\sigma,\tau;-1)$.*

Proof. Since $(Y_t)_{t\geqslant 0}$ is $QH(\chi_1,\chi_2;0,0;-1)$, formulas (1.2) and (1.1) are satisfied. Therefore, it is sufficient to check that the quadratic harness condition (1.4) holds with (1.7) and $q=-1$. Note that for $0\leqslant s<t<u$:

$$\begin{aligned}
\frac{(u-t)(u+t)}{(u-s)(u+s)} &= \frac{(u-t)(u(1+\sigma t)+\tau+t)}{(u-s)(u(1+\sigma s)+\tau+s)}+\frac{(u-t)(t-s)(\tau-\sigma u^2)}{(u-s)(u+s)(u(1+\sigma s)+\tau+s)},\\
\frac{(t-s)(t+s)}{(u-s)(u+s)} &= \frac{(t-s)(t(1+\sigma s)+\tau+s)}{(u-s)(u(1+\sigma s)+\tau+s)}-\frac{(u-t)(t-s)(\tau-\sigma s^2)}{(u-s)(u+s)(u(1+\sigma s)+\tau+s)},\\
\frac{(u-t)(t-s)(\chi_1 u-\chi_2)}{(u-s)(u+s)} &= \frac{(u-t)(t-s)(\eta u-\theta)}{(u-s)(u(1+\sigma s)+\tau+s)}-\frac{(u-t)(t-s)(\tau-\sigma u^2)(\chi_2+\chi_1 s)}{(u-s)(u+s)(u(1+\sigma s)+\tau+s)},\\
\frac{(u-t)(t-s)(\chi_2-\chi_1 s)}{(u-s)(u+s)} &= \frac{(u-t)(t-s)(\theta-\eta s)}{(u-s)(u(1+\sigma s)+\tau+s)}+\frac{(u-t)(t-s)(\tau-\sigma s^2)(\chi_2+\chi_1 u)}{(u-s)(u+s)(u(1+\sigma s)+\tau+s)},\\
\frac{(u-t)(t-s)}{u+s} &= \frac{(u-t)(t-s)}{u(1+\sigma s)\tau+s}+\frac{(u-t)(t-s)(\tau+\sigma su)}{(u+s)(u(1+\sigma s)+\tau+s)}.
\end{aligned}$$

So (8.3) used with χ_1 and χ_2 instead of η and θ, respectively, implies

$$\begin{aligned}\mathbb{E}(Y_t^2|\mathcal{F}_{s,u}) = & \tfrac{(u-t)(u(1+\sigma t)+\tau+t)}{(u-s)(u(1+\sigma s)+\tau+s)}Y_s^2 + \tfrac{(t-s)(t(1+\sigma s)+\tau+s)}{(u-s)(u(1+\sigma s)+\tau+s)}Y_u^2 \\ & + \tfrac{(u-t)(t-s)(\eta u-\theta)}{(u-s)(u(1+\sigma s)+\tau+s)}Y_s + \tfrac{(u-t)(t-s)(\theta-\eta s)}{(u-s)(u(1+\sigma s)+\tau+s)}Y_u + \tfrac{(u-t)(t-s)}{u(1+\sigma s)\tau+s} \\ & + \tfrac{(u-t)(t-s)(\tau+\sigma su)}{(u+s)(u(1+\sigma s)+\tau+s)} \\ & + \tfrac{(u-t)(t-s)(\tau-\sigma u^2)}{(u-s)(u+s)(u(1+\sigma s)+\tau+s)}(Y_s^2-(\chi_2+\chi_1 s)Y_s) \\ & - \tfrac{(u-t)(t-s)(\tau-\sigma s^2)}{(u-s)(u+s)(u(1+\sigma s)+\tau+s)}(Y_u^2-(\chi_2+\chi_1 u)Y_u).\end{aligned}$$

Using (8.1), we observe that the last three terms cancel out. Consequently, we conclude that $(Y_t)_{t\geqslant 0}$ is $QH(\eta,\theta;\sigma,\tau;-1)$. □

Therefore, all the results discussed in the previous section, with η and θ replaced by χ_1 and χ_2, respectively, hold for $QH(\eta,\theta;\sigma,\tau;-1)$.

Under assumption (8.9), a family of elements $\{\widetilde{\mathbb{P}}_{s,t}\}_{0\leqslant s<t}$ associated with $QH(\eta,\theta;\sigma,\tau;-1)$ satisfies Definition 1.3 in [24]. This is because we have only modified the parameters of the bi-Poisson process, preserving all the relationships discussed in Section 8.1. Therefore, the solution of the q-commutation equation actually corresponds to the pre-generator of the quadratic harness with $q=-1$.

So now we will use Theorem 1.6.1 and Theorem 3.3.1 to verify that we have the same formula for the infinitesimal generator as the one in (8.7). Under assumption (8.8),

$$\xi = 2\sqrt{1-\sigma\tau}, \qquad \xi_0 = \tfrac{\tau+2t+\sigma t^2}{1-\sigma\tau}, \qquad \xi_1 = \tfrac{1+\sigma t}{\sqrt{1-\sigma\tau}}-1, \qquad \xi_2 = \tfrac{\theta-\eta t}{\sqrt{1-\sigma\tau}}.$$

see (1.21) and (1.22). Moreover, $\widetilde{q}$, given in (1.24), is equal to -1. Because $[2n]_{\widetilde{q}}=0$ and $[n]_{\widetilde{q}}^2=[n]_{\widetilde{q}}$, $n\in\mathbb{N}_0$, we get the following three-step recurrence:

$$\begin{aligned}&\widetilde{W}_0(y;x,t)=1,\\ &\widetilde{W}_{n+1}(y;x,t) = \Big(y+(-1)^n x-(\eta\xi_0+\xi_2(1+\xi_1))\tfrac{1-(-1)^n}{2}\Big)\widetilde{W}_n(y;x,t), \quad n\geqslant 0.\end{aligned}$$

It is easy to check that $\eta\xi_0+\xi_2(1+\xi_1)=\chi_2+\chi_1 t$. According to [22, Theorem A.1], this implies that the polynomials $\{\widetilde{W}_n(\cdot;x,t)\}_{n\geqslant 0}$ are orthogonal with respect to the Dirac measure $\delta_{\chi_2+\chi_1 t-x}$. Therefore, we obtain formula (8.7) with η and θ replaced by χ_1 and χ_2, respectively.

8.3. Existence of $QH(\eta,\theta;\sigma,\tau;-1)$ when $1+\chi_1\chi_2<0$

In the previous section, we constructed a quadratic harness assuming (8.9). The question arises whether the quadratic harnesses exists for the remaining range of parameters, that is, when $1+\chi_1\chi_2<0$.

We consider only quadratic harnesses with all moments finite. As a consequence, martingale polynomials exist, although they may not be uniquely determined, as explained in Section 8.1.2. In particular, there is ambiguity when $q=-1$ because the coefficient at p_{n-1} vanishes for $n=2$ in the three-step recurrence, as shown in (1.12) and (1.13). Therefore, if the corresponding quadratic harness exists, the distribution of X_t must be supported on the zeros of the polynomial $p_2(\cdot;t)$ (given in (1.11)):

$$x_{t,\pm}=\tfrac{1}{2}\left(\chi_1 t+\chi_2\pm\sqrt{4t+(\chi_1 t+\chi_2)^2}\right).$$

Then, by (1.6),

$$\mathbb{V}\mathrm{ar}(X_t|X_s=x_{s,\pm})=\tfrac{t-s}{1+\sigma s}(1+\eta x_{s,\pm}+\sigma x_{s,\pm}^2)\xrightarrow{s\to 0^+} t\left(1+\tfrac{1}{2}\eta(\chi_2\pm|\chi_2|)+\tfrac{\sigma}{2}\chi_2(\chi_2\pm|\chi_2|)\right).$$

We will show that the above limiting expression leads to a contradiction when $1+\chi_1\chi_2<0$.

If $\chi_2\geqslant 0$, then the expression taken with plus signs is equal to

$$t(1+\eta\chi_2+\sigma\chi_2^2)=t(1+\chi_1\chi_2)<0.$$

Since the conditional variance is nonnegative almost surely, this implies that $\mathbb{P}(X_s=x_{s,+})=0$ for all small s, and hence $\mathbb{P}(X_s=x_{s,-})=1$. As a result,

$$\mathbb{E}X_s=x_{s,-}=\frac{-2s}{\chi_1 s+\chi_2+\sqrt{4s+(\chi_1 s+\chi_2)^2}}<0.$$

The last inequality holds since $\sqrt{4s+(\chi_1 s+\chi_2)^2}>\sqrt{(\chi_1 s+\chi_2)^2}\geqslant-(\chi_1 s+\chi_2)$. This contradicts (1.2).

Similarly, the case of $\chi_2<0$ leads to an analogous contradiction.

Consequently, we can conclude that there exist no quadratic harnesses $QH(\eta,\theta;\sigma,\tau;-1)$ when $1+\chi_1\chi_2<0$.

Chapter 9

Classical quadratic harnesses

In this chapter, we focus on classical quadratic harnesses $QH(\eta, \theta; \sigma, \tau; 1 - 2\sqrt{\sigma\tau})$. The term 'classical' is justified by the fact that when $\sigma\tau = 0$, quadratic harnesses become some well-known Lévy processes such as the Wiener or the standardized Poisson process (see [21, Section 4.2]).

The first part of this chapter discusses the results of Theorems 1.6.1 and 3.3.2 when $q = 1 - 2\sqrt{\sigma\tau}$. In the second part, we derive an alternative representation of the pre-generator that does not rely on orthogonal polynomials. Furthermore, we compare the obtained formula with some results known from the literature, specifically for Lévy processes and bi-Poisson processes.

9.1. Infinitesimal generator through the orthogonal polynomials

We continue to work under assumptions (1.5) and (1.9), which reduce to

$$\sigma, \tau \geqslant 0, \qquad \sigma\tau < 1 \tag{9.1}$$

when $q = 1 - 2\sqrt{\sigma\tau}$. Moreover, after simplification, we obtain

$$\xi = 2(1 - \sqrt{\sigma\tau}), \qquad \xi_0 = \tfrac{(\sqrt{\tau}+\sqrt{\sigma}t)^2}{(1-\sqrt{\sigma\tau})^2}, \qquad \xi_1 = \tfrac{\sigma t+\sqrt{\sigma\tau}}{1-\sqrt{\sigma\tau}}, \qquad \xi_2 = \tfrac{\theta-\eta t}{1-\sqrt{\sigma\tau}}, \qquad \widetilde{q} = 1,$$

see (1.21), (1.22) and (1.24). Consequently, $[n]_{\widetilde{q}} = n$ for all $n \in \mathbb{N}_0$.
When $\sigma t^2 = \tau = 0$, then $\xi_0 = 0$ and consequently $\widetilde{b}_n = 0$ for all $n \in \mathbb{N}_0$. Then according to Theorem A.1 in [22], the polynomials $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$ are orthogonal with respect to the Dirac measure concentrated at $x + \theta - \eta t$, which is a root of $\widetilde{W}_1(\cdot; x, t)$. As a result,

we obtain:

$$(\mathbf{A}_t f)(x) = \begin{cases} \frac{1+\eta x+\sigma x^2}{2} f''(x) & \text{when } \theta = \eta t, \\ (1+\eta x+\sigma x^2)\frac{f(x+\theta-\eta t)-f(x)-f'(x)(\theta-\eta t)}{(\theta-\eta t)^2} & \text{when } \theta \neq \eta t, \end{cases} \tag{9.2}$$

for any polynomial (Theorem 1.6.1) and any bounded function with a bounded and continuous second derivative (Theorem 3.3.2). In particular, when $\sigma = \tau = \theta = \eta = 0$, we get the infinitesimal generator of the Wiener process, compare with Example 1.6.3.

As the next example shows the domain of $\mathbf{A}_t$ for the standardized Poisson process is larger than the one appearing in Theorem 3.3.2.

Example 9.1.1. *Recall that the standardized Poisson process* $(Y_t)_{t\geqslant 0}$ *defined in Example 1.2.3 is* $QH\left(0, \frac{1}{\sqrt{\lambda}}; 0, 0; 1\right)$, $\lambda > 0$.
We will derive the infinitesimal generator of $(Y_t)_{t\geqslant 0}$ *directly from the definition. In order to do so, note that for* $t \geqslant 0$ *and* $h > 0$ *we have*

$$Y_{t+h} = Y_t + \frac{N_{t+h}-N_t-\lambda h}{\sqrt{\lambda}}$$

and the summands on the right-hand side are independent since $(N_t)_{t\geqslant 0}$ *is a classical Poisson process with parameter* $\lambda > 0$. *Consequently, for any bounded function* f *on* $\mathbb{R}$, *we have*

$$\begin{aligned}\mathbb{E}(f(Y_{t+h})|Y_t = x) - f(x) &= \mathbb{E} f\left(\frac{N_{t+h}-N_t-\lambda h}{\sqrt{\lambda}} + x\right) - f(x) \\ &= \sum_{k=0}^{\infty}\left(f\left(\frac{k-\lambda h}{\sqrt{\lambda}} + x\right) - f(x)\right)\frac{(\lambda h)^k}{k!}e^{-\lambda h},\end{aligned}$$

where in the last step we used the fact that $N_{t+h} - N_t$ *has a Poisson distribution with parameter* λh. *Assume that* f *is differentiable (thus continuous). Then we have*

$$\lim_{h\to 0^+} \frac{f\left(x-\frac{\lambda h}{\sqrt{\lambda}}\right)-f(x)}{h} e^{-\lambda h} = -\sqrt{\lambda} f'(x)$$

and

$$\lim_{h\to 0^+} \frac{f\left(x+\frac{1-\lambda h}{\sqrt{\lambda}}\right)-f(x)}{h}\lambda h e^{-\lambda h} = \lambda\left(f(x+\tfrac{1}{\sqrt{\lambda}}) - f(x)\right).$$

Furthermore,

$$\lim_{h\to 0^+}\frac{1}{h}\left|\sum_{k=2}^{\infty}\left(f\left(\frac{k-\lambda h}{\sqrt{\lambda}}+x\right)-f(x)\right)\frac{(\lambda h)^k}{k!}e^{-\lambda h}\right| \leqslant 2\sup_{x\in\mathbb{R}}|f(x)|\cdot\lim_{h\to 0^+}e^{-\lambda h}\frac{e^{\lambda h}-1-h}{h}=0.$$

As a result, the limit of the right-hand side of (1.15) *exists. Hence* $f\in\mathcal{D}(\mathbf{A}_t^+)$ *for* $t\geqslant 0$, *and*

$$(\mathbf{A}_t^+f)(x)=\lambda\left(f(x+\tfrac{1}{\sqrt{\lambda}})-f(x)\right)-\sqrt{\lambda}f'(x).$$

The above formula coincides with the one from (9.2), *however, in the above reasoning, there was no need to assume that* f *has a bounded second derivative.*
A similar result can be obtained for $\mathbf{A}_t^-$.

If $\sigma t^2>0$ or $\tau>0$, then $\xi_0>0$ and $\xi_1\geqslant 0$. Consequently, since the polynomials $\{\widetilde{W}_n(\cdot;x,t)\}_{n=0}^\infty$ are orthogonal with respect to some probability measure when $1+\eta x+\sigma x^2>0$ (and the corresponding moment functional is non-negative definite), the following conditions hold:

$$\prod_{k=1}^{N}[1+\widetilde{\gamma}_k(x)(\sigma\widetilde{\gamma}_k(x)+\eta)]\geqslant 0,\qquad N\in\mathbb{N},$$

compare with identity (A.5).
In particular, as t and x approach zero, the above formula for $N=1$ implies that the classical quadratic harnesses $QH(\eta,\theta;\sigma,\tau;1-2\sqrt{\sigma\tau})$ with all moments finite can exist only if

$$1+\chi_1\chi_2\geqslant 0, \tag{9.3}$$

where χ_1 and χ_2 are given in (7.2) (recall the analogous reasoning from page 90).
The construction of a classical quadratic harness $QH(\eta,\theta;\sigma,\tau;1-2\sqrt{\sigma\tau})$ was done only for some parameters in [16], but not for the full admissible range of parameters given in (9.3). Nevertheless, a three-step recurrence for martingale polynomials under assumption (9.3) is known ([18, Proposition 4.4]).

9.2. Infinitesimal generator through the cotangent function

In this section, we aim to derive an alternative, algebraic representation of the infinitesimal generator of $QH(\eta,\theta;\sigma,\tau;1-2\sqrt{\sigma\tau})$ that differs from the one obtained in Theorem 1.6.1. To achieve this, let us recall the algebra $\mathcal{Q}$ along with its subspaces $\mathcal{Q}_i$, $i \in \mathbb{N}_0$, and the elements $\mathbb{E}$, $\mathbb{F}$ and $\mathbb{D}_1$ defined in (2.2), (2.3) and (2.6), respectively. Recall that in the algebraic approach, the main objective is to find a solution of the q-commutation equation (2.11) that satisfies the initial condition (2.12). In the following, we will present this solution when $q = 1 - 2\sqrt{\sigma\tau}$ without making use of orthogonal polynomials.

For this purpose, let us consider the following function of a complex variable $z \in \mathbb{C}$:

$$f(z) := \sqrt{\tfrac{z}{2}} \cot\left(\sqrt{\tfrac{z}{2}}\right), \tag{9.4}$$

which is well-defined in a neighborhood $\mathcal{U}$ of zero. Note that we do not need to be precise about the branch of the square root function in (9.4), since there are only even powers of z in the Taylor expansion of $z\cot(z)$. Moreover, f is analytic in $\mathcal{U}$.

Furthermore, we can check, after some calculations, that for all $z \in \mathcal{U}$:

$$f(z)(f(z) - 1) + 2zf'(z) + \tfrac{1}{2}z = 0, \tag{9.5}$$

$$2f'(z)(f(z) - 1) + 3f'(z) + 2zf''(z) + \tfrac{1}{2} = 0. \tag{9.6}$$

In order to apply f to elements of $\mathcal{Q}$, we introduce the notation:

$$f(\mathbb{G}) := \sum_{k=0}^{\infty} d_k \mathbb{G}^k$$

where $\{d_k\}_{k=0}^{\infty}$ are the coefficients of the Taylor expansion of f around zero; it is important to observe that this expression gives a well-defined element of $\mathcal{Q}$ when $\mathbb{G} \in \mathcal{Q}$ is such that in each coordinate the series has only a finite number of summands.

Moreover, using L'Hôpital's rule, we can find the first coefficient d_0:

$$d_0 = \lim_{z\to 0} f(z) = \lim_{z\to 0} \cos^2\left(\sqrt{\tfrac{z}{2}}\right) = 1. \tag{9.7}$$

Let us define a function of g, analytic in the neighbourhood $\mathcal{U}$:

$$g(z) := \tfrac{f(z)-1}{z} = \sum_{k=1}^{\infty} d_k z^{k-1} \tag{9.8}$$

(so g is the zeroth Jackson derivative of f).
As before, we will apply g to elements $\mathbb{G} \in \mathcal{Q}$ for which $\sum_{k=1}^{\infty} d_k \mathbb{G}^{k-1}$ consists of only finite sums in each coordinate.
The following element $\mathbb{G} \in \mathcal{Q}$ will play an important role in solving the q-commutation equation when $q = 1 - 2\sqrt{\sigma\tau}$:

$$\mathbb{G} := \varepsilon f(\mathbb{C}) + \delta \mathbb{F}\mathbb{D}_1 + \gamma \mathbb{D}_1 \tag{9.9}$$

with $\mathbb{C} := \mathbb{D}_1\mathbb{N}$, $\mathbb{N} := \alpha(2\mathbb{F}\mathbb{D}_1 - \mathbb{E}) + \beta\mathbb{D}_1$ for some real coefficients α, β, δ, γ and ε to be specified later. Here and in the remainder of this section, f denotes the function given by (9.4). The element $\mathbb{G}$ is well-defined since $\mathbb{C}$ is a sum of certain elements from $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and consequently, the series $f(\mathbb{C})$ consists of finite sums coordinate-wise.

Lemma 9.2.1. *Element $\mathbb{G} \in \mathcal{Q}$ satisfies the following identities:*

$$\begin{aligned} \mathbb{G}\mathbb{N} &= \varepsilon \mathbb{N} f(\mathbb{C}) + 2\alpha\varepsilon f'(\mathbb{C})\mathbb{C} + \delta\mathbb{F}\mathbb{C} + \gamma\mathbb{C}, \\ \mathbb{F}\mathbb{G} - \mathbb{G}\mathbb{F} &= \varepsilon\mathbb{N} f(\mathbb{C}) g(\mathbb{C}) + 2\alpha\varepsilon f'(\mathbb{C})\mathbb{C} g(\mathbb{C}) + \alpha\varepsilon\mathbb{F}\mathbb{D}_1 + \tfrac{1}{2}\beta\varepsilon\mathbb{D}_1 - \delta\mathbb{F} - \gamma\mathbb{E}, \end{aligned}$$

where g is given by (9.8).

Proof. We begin with the proof of the first identity. Note that (5.3) when $q = 1$ gives

$$\mathbb{C} = \mathbb{D}_1\mathbb{N} = (\alpha(2\mathbb{D}_1\mathbb{F} - \mathbb{E}) + \beta\mathbb{D}_1)\mathbb{D}_1 = \mathbb{N}\mathbb{D}_1 + 2\alpha\mathbb{D}_1.$$

Multiplying both sides by $\mathbb{N}$ from the right, we get that $\mathbb{C}\mathbb{N} = \mathbb{N}\mathbb{C} + 2\alpha\mathbb{C}$. Using a simple induction, we obtain:

$$\mathbb{C}^k\mathbb{N} = \mathbb{N}\mathbb{C}^k + 2\alpha k \mathbb{C}^k, \qquad k \in \mathbb{N}_0. \tag{9.10}$$

Consequently, by the definition of $\mathbb{G}$ we finally get

$$\mathbb{G}\mathbb{N} = \varepsilon\mathbb{N} f(\mathbb{C}) + 2\alpha\varepsilon \sum_{k=0}^{\infty} k d_k \mathbb{C}^k + \delta\mathbb{F}\mathbb{D}_1\mathbb{N} + \gamma\mathbb{D}_1\mathbb{N},$$

which immediately implies the first identity.
In order to prove the second identity, observe that formula (5.3) for $q = 1$ gives

$$\mathbb{N}\mathbb{F} - \mathbb{F}\mathbb{N} = 2\alpha\mathbb{F}(\mathbb{D}_1\mathbb{F} - \mathbb{F}\mathbb{D}_1) + \beta(\mathbb{D}_1\mathbb{F} - \mathbb{F}\mathbb{D}_1) = 2\alpha\mathbb{F} + \beta\mathbb{E}.$$

Multiplying by $\mathbb{D}_1$ from the left yields $\mathbb{C}\mathbb{F}-(\mathbb{F}\mathbb{D}_1+\mathbb{E})\mathbb{N} = 2\alpha(\mathbb{F}\mathbb{D}_1+\mathbb{E})+\beta\mathbb{D}_1$. Consequently,

$$\mathbb{C}\mathbb{F} - \mathbb{F}\mathbb{C} = 2\mathbb{N} + 3\alpha\mathbb{E}$$

and using induction together with (9.10), we can show:

$$\mathbb{C}^k\mathbb{F} - \mathbb{F}\mathbb{C}^k = 2k\mathbb{N}\mathbb{C}^{k-1} + \alpha k(2k+1)\mathbb{C}^{k-1}, \qquad k \in \mathbb{N}.$$

Hence, by (5.3), we finally obtain

$$\begin{aligned}\mathbb{G}\mathbb{F} - \mathbb{F}\mathbb{G} &= 2\varepsilon\mathbb{N}\sum_{k=1}^{\infty} kd_k\mathbb{C}^{k-1} + 2\alpha\varepsilon\sum_{k=1}^{\infty} k(k-1)d_k\mathbb{C}^{k-1} + 3\alpha\varepsilon\sum_{k=1}^{\infty} kd_k\mathbb{C}^{k-1} + \delta\mathbb{F} + \gamma\mathbb{E}\\ &= 2\varepsilon\mathbb{N}f'(\mathbb{C}) + 2\alpha\varepsilon\mathbb{C}f''(\mathbb{C}) + 3\alpha\varepsilon f'(\mathbb{C}) + \delta\mathbb{F} + \gamma\mathbb{E}.\end{aligned}$$

From (9.5), we get $f(z)g(z)+2f'(z)+\frac{1}{2} = 0$, which in terms of elements of $\mathcal{Q}$ is understood as $f(\mathbb{C})g(\mathbb{C}) + 2f'(\mathbb{C}) + \frac{1}{2}\mathbb{E} = \mathbb{0}$. This, together with formula (9.6) and the definition of $\mathbb{N}$, proves the second identity. □

Now we specify the parameters α, β, γ, δ, and ε appearing in (9.9). We do so to establish a close relationship between the element $\mathbb{G}$ and the pre-generator of quadratic harness $QH(\eta,\theta;\sigma,\tau;q)$ when $q = 1-2\sqrt{\sigma\tau}$. Namely, we set the parameters as follows:

$$\begin{aligned}&\alpha := \tfrac{\theta\sigma t+\eta\tau+\sqrt{\sigma\tau}(\theta+\eta t)}{(1-\sqrt{\sigma\tau})^2}, \qquad \beta := 2\tfrac{(\sqrt{\tau}+\sqrt{\sigma}t)^2-\gamma^2}{(1-\sqrt{\sigma\tau})^2}, \qquad \gamma := -\tfrac{\theta-\eta t}{2},\\ &\delta := \sigma t + \sqrt{\sigma\tau}, \qquad \varepsilon := 1-\sqrt{\sigma\tau}.\end{aligned} \tag{9.11}$$

Under assumptions (9.1), $\delta \geqslant 0$ and $\varepsilon > 0$ for $t \geqslant 0$. Moreover, it is easy to check that the following identities are satisfied:

$$\begin{aligned}&\varepsilon-\delta = q-\sigma t, \qquad \tau + (1-q)t + \sigma t^2 = (\sqrt{\tau}+\sqrt{\sigma}t)^2, \qquad \delta^2 = \sigma(\sqrt{\tau}+\sqrt{\sigma}t)^2,\\ &\gamma^2 + \tfrac{1}{2}\beta\varepsilon^2 = (\sqrt{\tau}+\sqrt{\sigma}t)^2, \qquad 2\gamma\delta + \alpha\varepsilon^2 = \eta(\sqrt{\tau}+\sqrt{\sigma}t)^2.\end{aligned} \tag{9.12}$$

Now we are in a position to prove the following formula for the pre-generator of $QH(\eta,\theta;\sigma,\tau;1-2\sqrt{\sigma\tau})$:

Theorem 9.2.2. *Let us consider $\mathbb{G}$ as given in (9.9) with parameters (9.11), where $0 \leqslant \sigma\tau < 1$. Then the pre-generator of $QH(\eta,\theta;\sigma,\tau;1-2\sqrt{\sigma\tau})$ at time $t \geqslant 0$ can* *be represented as*

$$\mathbb{H}_t = (\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathbb{D}_1\mathbb{G}^{-1}. \tag{9.13}$$

Proof. We will show that $(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathbb{D}_1\mathbb{G}^{-1}$ satisfies (2.11) with (2.12). So, by the uniqueness of the solution of the q-commutation equation, we will get the desired result. Firstly, note that $\mathbb{G}$ is a sum of certain elements from $\mathcal{Q}_k$, $k \in \mathbb{N}_0$. The summand of $\mathbb{G}$ coming from $\mathcal{Q}_0$ is equal to $\varepsilon\mathbb{E}+\delta\mathbb{F}\mathbb{D}_1$, see (9.7). Since $\varepsilon > 0$ and $\delta \geqslant 0$, the nth coordinate of $\varepsilon\mathbb{E} + \delta\mathbb{F}\mathbb{D}_1$ is a monomial of degree n with nonzero leading coefficients equal to $\varepsilon + \delta n$. Hence $\mathbb{G}$ is invertible due to Remark 4.1.5 and $(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathbb{D}_1\mathbb{G}^{-1}$ is well-defined.

To see that (2.12) is satisfied, observe first that Lemma 4.1.3 yields that $\mathbb{G}(\mathbb{E} - \mathbb{F}\mathbb{D}) = (\varepsilon\mathbb{E} + \delta\mathbb{F}\mathbb{D}_1)(\mathbb{E} - \mathbb{F}\mathbb{D}) = \varepsilon(\mathbb{E} - \mathbb{F}\mathbb{D})$, see (2.9). Consequently, in view of (2.9),

$$(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathbb{D}_1\mathbb{G}^{-1}(\mathbb{E} - \mathbb{F}\mathbb{D}) = \tfrac{1}{\varepsilon}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathbb{D}_1(\mathbb{E} - \mathbb{F}\mathbb{D}) = \mathbb{0}.$$

As a result, the initial condition (2.12) is satisfied.

In order to prove (2.11), it is enough to show

$$\begin{aligned}(1 + \sigma t)\mathbb{D}_1\mathbb{G}^{-1}\mathbb{F} - (q - \sigma t)\mathbb{F}\mathbb{D}_1\mathbb{G}^{-1} &= \mathbb{E} + (\theta - \eta t)\mathbb{D}_1\mathbb{G}^{-1} \\ &\quad + (\tau + (1-q)t + \sigma t^2)\mathbb{D}_1\mathbb{G}^{-1}(\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2)\mathbb{D}_1\mathbb{G}^{-1},\end{aligned}$$

since multiplying the latter from the left by $\mathbb{E} + \eta\mathbb{F} + \sigma\mathbb{F}^2$ gives (2.11) with $\mathbb{H}_t$ given by (9.13).

Since $\mathbb{G}$ is invertible, we equivalently need to show

$$\mathbb{D}_1\mathbb{G}^{-1}((1+\sigma t)\mathbb{F}\mathbb{G}-(\tau+(1-q)t+\sigma t^2)(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\mathbb{D}_1) = (q-\sigma t)\mathbb{F}\mathbb{D}_1+\mathbb{G}+(\theta-\eta t)\mathbb{D}_1. \tag{9.14}$$

Due to the definition (9.8) of g, we see that $f(\mathbb{C}) = \mathbb{C}g(\mathbb{C}) + \mathbb{E}$, and thus the right-hand side of the above formula is equal to

$$\begin{aligned}&\varepsilon\mathbb{E} + (\delta + q - \sigma t)\mathbb{F}\mathbb{D}_1 + \varepsilon\mathbb{C}g(\mathbb{C}) + (\theta - \eta t + \gamma)\mathbb{D}_1 \\ &\qquad = \varepsilon\mathbb{E} + (\delta + q - \sigma t)\mathbb{F}\mathbb{D}_1 + (\theta - \eta t + \gamma)\mathbb{D}_1 \\ &\qquad + \varepsilon\mathbb{D}_1\mathbb{G}^{-1}(\varepsilon\mathbb{N}f(\mathbb{C}) + 2\alpha\varepsilon f'(\mathbb{C})\mathbb{C} + \delta\mathbb{F}\mathbb{C} + \gamma\mathbb{C})g(\mathbb{C}),\end{aligned}$$

where in the last equality we used the first formula from Lemma 9.2.1. Next the second formula from the same lemma yields that the right-hand side of (9.14) is equal to

$$\begin{aligned}&\varepsilon(\mathbb{E}-\mathbb{D}_1\mathbb{F})+(\delta+q-\sigma t)\mathbb{F}\mathbb{D}_1+(\theta-\eta t+\gamma)\mathbb{D}_1\\&\qquad+\varepsilon\mathbb{D}_1\mathbb{G}^{-1}(\mathbb{F}\mathbb{G}-\alpha\varepsilon\mathbb{F}\mathbb{D}_1-\tfrac{1}{2}\beta\varepsilon\mathbb{D}_1+\delta\mathbb{F}f(\mathbb{C})+\gamma f(\mathbb{C})),\end{aligned}$$

where we have used $f(\mathbb{C})=\mathbb{C}g(\mathbb{C})+\mathbb{E}$ again. The definition of $\mathbb{C}$ and identity (5.3) imply

$$\begin{aligned}&(\delta+q-\sigma t-\varepsilon)\mathbb{F}\mathbb{D}_1+(\theta-\eta t+2\gamma)\mathbb{D}_1\\&\qquad+\mathbb{D}_1\mathbb{G}^{-1}((\varepsilon+\delta)\mathbb{F}\mathbb{G}-\alpha\varepsilon^2\mathbb{F}\mathbb{D}_1-\tfrac{1}{2}\beta\varepsilon^2\mathbb{D}_1-\delta\mathbb{F}(\delta\mathbb{F}\mathbb{D}_1+\gamma\mathbb{D}_1)-\gamma(\delta\mathbb{F}\mathbb{D}_1+\gamma\mathbb{D}_1)).\end{aligned}$$

Recalling how the parameters α, β, γ, δ, and ε were specified in (9.12), we get that the right-hand side of (9.14) simplifies to:

$$\mathbb{D}_1\mathbb{G}^{-1}((1+\sigma t)\mathbb{F}\mathbb{G}-(\tau+(1-q)t+\sigma t^2)(\mathbb{E}+\eta\mathbb{F}+\sigma\mathbb{F}^2)\mathbb{D}_1)$$

and this is exactly the left-hand side of (9.14), which is what we were supposed to show.

□

While the pre-generator $\mathbb{H}_t$ has a simple form, finding a similarly simple formula for the generator $\mathbb{A}_t$ seems to be challenging. We have only succeeded for $\alpha=\delta=0$, as it will be shown later. Recall, however, that the algebraic representation (2.13) remains valid in general, without the restriction $\alpha=\delta=0$.

We will now show that Theorem 9.2.2 is consistent with the results known previously from the literature.

Example 9.2.3 (Infinitesimal generators of Lévy processes $QH(0,\theta;0,\tau;1)$)**.** *In this case, we get $\alpha=\delta=0$, which implies that $\mathbb{H}_t$ is in fact a function of $\mathbb{D}_1$ only. Specifically,*

$$\mathbb{H}_t=\widetilde{h}(\mathbb{D}_1), \tag{9.15}$$

where $\widetilde{h}(z):=\frac{z}{f(-2(\gamma^2-\tau)z^2)+\gamma z}$ is well-defined in some neighborhood of zero.

Taking the principal square root function in the definition (9.4) of f, we can write

$$f\left(-2(\gamma^2-\tau)z^2\right)=iz\sqrt{\gamma^2-\tau}\cot(iz\sqrt{\gamma^2-\tau}),$$

where i is the imaginary unit. The fact that we have the same z on both sides of the above equation and that we do not need to care for its argument, follows from the identity:

$$-z\cot(-z) = z\cot(z),$$

which holds in a neighborhood of zero.

Moreover, $\widetilde{h}$ is an analytic function in its domain. Indeed, let $a := \theta^2 - 4\tau$. Then the well-known relation between coth *and* cot *implies*

$$\widetilde{h}(z) = \begin{cases} \frac{z}{1+\gamma z} & \text{when } \gamma^2 = \tau, \\ \frac{z}{iz\sqrt{\gamma^2-\tau}\cot(iz\sqrt{\gamma^2-\tau})+\gamma z} & \text{when } \gamma^2 \neq \tau \end{cases} = \begin{cases} \frac{2z}{2+\theta z} & \text{when } a = 0, \\ \frac{2}{\sqrt{a}\coth(\sqrt{a}z/2)+\theta} & \text{when } a > 0, \\ \\ \frac{2}{\sqrt{-a}\cot(\sqrt{-a}z/2)+\theta} & \text{when } a < 0 \end{cases}$$

and the functions appearing on the right-hand side are analytic in the neighborhood of zero.

In [24, Section 5] it was shown that if $\mathbb{H}_t$ is of the form (9.15)*, then $\mathbb{A}_t$ can be represented as*

$$\mathbb{A}_t = \widetilde{H}(\mathbb{D}_1), \tag{9.16}$$

where $\widetilde{H}$ is the antiderivative of $\widetilde{h}$ such that $\widetilde{H}(0) = 0$.

The antiderivative $\widetilde{H}$ takes the following forms:

a) if $\theta^2 = 4\tau = 0$ (Wiener process):

$$\widetilde{H}(z) = \tfrac{1}{2}z^2,$$

b) if $\theta^2 = 4\tau > 0$ (Gamma type process):

$$\widetilde{H}(z) = \tfrac{2}{\theta}z - \tfrac{4}{\theta^2}\ln(1+\theta z/2),$$

c) if $\theta^2 > 4\tau = 0$ (Poisson type process):

$$\widetilde{H}(z) = \tfrac{1}{\theta}z - \tfrac{1}{\theta^2} + \tfrac{1}{\theta^2}e^{-\theta z},$$

d) if $\theta^2 > 4\tau > 0$ (Pascal type process):

$$\widetilde{H}(z) = -\tfrac{1}{\tau}\ln\left(\tfrac{1}{2}(1+\tfrac{\theta}{\sqrt{a}})\exp(-\tfrac{\theta-\sqrt{a}}{2}z) + \tfrac{1}{2}(1-\tfrac{\theta}{\sqrt{a}})\exp(-\tfrac{\theta+\sqrt{a}}{2}z)\right),$$

e) if $\theta^2 < 4\tau$ (Meixner type process).

$$\widetilde{H}(z) = \tfrac{\theta}{2\tau}z - \tfrac{1}{\tau}\ln(\cos(\sqrt{-a}z/2) + \tfrac{\theta}{\sqrt{-a}}\sin(\sqrt{-a}z/2)),$$

If $\sigma = \eta = 0$, then $QH(0,\theta;0,\tau;1)$ is a Lévy process, compare with [21, Remark 3.2]. Hence, the infinitesimal generator of such a process is well-known and does not depend on t. Moreover, it turns out that the infinitesimal generator can be written as

$$\mathbf{A}_t = \mathbf{A} = \psi(-i\partial_x), \tag{9.17}$$

see [5, Section 3], where ψ is a cumulant generating function of the considered Lévy process, more details can be found in [39, Theorem 2]. The right-hand side of the above expression should be understood as

$$\sum_{k=1}^{\infty} c_k\partial_x^k,$$

where $\sum_{k=1}^{\infty} c_k(iz)^k$ is a Taylor expansion of ψ at 0 and ∂_x^k is an operator taking the kth derivative with respect to x.

Formulas for ψ for quadratic harnesses $QH(0,\theta;0,\tau;1)$ are well-known in the literature and can be obtained by taking the logarithm of the expressions in [21, Theorem 4.2] and setting $t = 1$ (since the distribution of X_1 is considered in [5]). These steps lead to the conclusion that

$$\widetilde{H}(z) = \psi(iz).$$

Furthermore, using the definition (2.1) of multiplication, it can be easily verified that the nth coordinate of (9.16) is indeed equal to $\mathbf{A}_t(x^n)$ as given in (9.17).

Example 9.2.4 (Infinitesimal generators for bi-Poisson processes $QH(\eta,\theta;0,0;1)$)**.** *In this case, we have:*

$$\mathbb{G} = f(-2\gamma^2\mathbb{D}_1^2) + \gamma\mathbb{D}_1.$$

If $\gamma = 0$*, then* $\mathbb{G} = f(\mathbb{0}) = \mathbb{E}$*. Consequently,*

$$\mathbb{H}_t = (\mathbb{E} + \eta\mathbb{F})\mathbb{D}_1. \tag{9.18}$$

If $\gamma \neq 0$*, the relations for the hyperbolic functions imply*

$$h(z) := f(-2z^2) + z = iz\cot(iz) + z = z(i\cot(iz) + 1) = z(\coth(z) + 1).$$

Note that the first equality is derived using the same arguments as discussed in the previous example. Set

$$\widetilde{h}(z) := \tfrac{z}{\gamma h(z)} = \tfrac{1}{\gamma(\coth(z)+1)} = \tfrac{1-\exp(-2z)}{2\gamma}.$$

Consequently,

$$\mathbb{H}_t = (\mathbb{E}+\eta\mathbb{F})\widetilde{h}(\gamma\mathbb{D}_1) = \tfrac{1}{2\gamma}(\mathbb{E}+\eta\mathbb{F})(\mathbb{E}-\exp(-2\gamma\mathbb{D}_1)) = \tfrac{1}{\theta-\eta t}(\mathbb{E}+\eta\mathbb{F})(\exp((\theta-\eta t)\mathbb{D}_1)-\mathbb{E}),$$

where the last step holds due to the formula for γ*, see* (9.11)*. The above formula is consistent with the result derived in [24, Section 5.2]. Furthermore, the limit of the above expression, when* γ *approaches zero, exists and coincides with* (9.18)*.*

Chapter 10

Discussion of the results

The algebraic approach presented in [24] provided a unified framework for an analysis of quadratic harnesses, regardless of their distributions. The key assumption in this approach is the finiteness of all moments, which is also held throughout this thesis. Importantly, this assumption is not restrictive and often leads to the unique determination of the process. Another assumption made in [24] is the requirement of infinite state space. However, as shown in Chapter 8, this assumption is unnecessary.

By considering associated polynomials instead of transition probabilities, the algebraic approach has allowed a systematization and simplification of previously used concepts, eliminating the need to refer to their complicated formulas, such as those for the martingale polynomials. Moreover, this approach helps to emphasize the role of the parameters of quadratic harnesses. Specifically, the influence of these parameters can be observed in the q-commutation equation, where, for example, setting parameters $\sigma = \tau = 0$ greatly simplifies the equation under consideration.

In a general case, finding a solution to the q-commutation equation without referring to the special family of orthogonal polynomials associated with the infinitesimal generator can be challenging. However, the formulas obtained in Theorem 1.6.1 and Theorem 3.3.1 are straightforward to apply if the form of the moment functional or the orthogonality measure is known. In many cases, the polynomials $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$ are related to Askey-Wilson polynomials, for which orthogonality measures are well-described in the literature (see Section 5 in [44]). Additionally, the orthogonality measure for $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$ simplifies to a Dirac measure in many interesting cases, as shown in Chapter 8 and Chapter 9.

It should also be emphasized that the results obtained in the thesis do not cover the entire domain of the infinitesimal generator. As shown in Chapters 8 and 9, starting directly from the definitions (1.15) and (1.16), we can derive formulas for the infinitesimal generator that apply to a larger domain than the one stated in Theorem 3.3.1. It is well-known that a domain of the infinitesimal generator, not just a formula itself, plays a crucial role in the analysis of a stochastic process. For example, the formulas for the infinitesimal generators of the Wiener process and the absolute value of the Wiener process are the same, although the domains are different.

Moreover, supports of quadratic harnesses are generally time-dependent, which has an impact on the domains of their infinitesimal generators. For instance, in the case of the standard Poisson process, it is not necessary to assume differentiability of domain's elements, while for the standardized Poisson process, which belongs to the family of quadratic harnesses, differentiability is required (see Example 9.1.1).

Results from the literature concerning infinitesimal generators cannot be directly applied to quadratic harnesses. Not only do we have to deal with inhomogeneous Markov processes, but the main problem is the time dependency of the supports. In many interesting cases of quadratic harnesses, $\mathrm{supp}(X_t)$ is bounded for each $t \geqslant 0$, but it is not uniformly bounded in t, i.e. there is no $M > 0$ such that $\mathrm{supp}(X_t) \subseteq [-M, M]$ for all $t \geqslant 0$. Therefore, in such a situation, it is not clear how to follow the classical scheme where the domain of the infinitesimal generator is a subset of some Banach space, and one can consider not only pointwise convergence but also convergence in the norm. Finding a suitable Banach space in which convergence for polynomials would imply appropriate convergence for the corresponding class of functions, seems to be a challenging task.

Despite these challenges, we have obtained some novel results which were previously unknown in the literature. We have derived the explicit and easily applicable formula for the infinitesimal generator of quadratic harnesses for a large range of their parameters. Generally, obtaining such a formula directly from the definition seems to be a very hard task due to a complicated description of transition probabilities. Additionally, we have identified a relatively wide class of functions that belong to the domain of the infinitesimal generator.

Appendix A

Orthogonal polynomials

This brief summary of some basic facts on orthogonal polynomials is based on [27]. Throughout this appendix, we consider polynomials with complex coefficients in one real variable x.

Definition A.0.1. Let $\{\mu_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and $\mathcal{L}$ be a complex-valued linear operator on the vector space of all polynomials, satisfying

$$\mathcal{L}[x^n] = \mu_n, \qquad n \geqslant 0.$$

Then $\mathcal{L}$ is called the *moment functional* determined by the *moment sequence* $\{\mu_n\}_{n=0}^{\infty}$. By linearity of $\mathcal{L}$, for any polynomial $f(x) = \sum\limits_{k=0}^{n} c_k x^k$ we have

$$\mathcal{L}[f(x)] = \sum_{k=0}^{n} c_k \mu_k. \tag{A.1}$$

Definition A.0.2. A sequence $\{P_n(x)\}_{n=0}^{\infty}$ is called an *orthogonal polynomial sequence* with respect to a moment functional $\mathcal{L}$ if two conditions hold:

1. $P_n(x)$ is a polynomial of degree n,
2. $\mathcal{L}[P_n(x)P_m(x)] = \chi_n \mathbb{1}_{\{m=n\}}$,

where $\chi_n \neq 0$ for all $n \geqslant 0$.

In the case we are interested in, the second condition of the above definition is too strong. Hence, we have to introduce a less restrictive definition.

Definition A.0.3. A sequence $\{P_n(x)\}_{n=0}^{\infty}$ is called a *weak orthogonal polynomial sequence* with respect to a moment functional $\mathcal{L}$ if two conditions hold:

1. $P_n(x)$ is a polynomial of degree n,
2. $\mathcal{L}[P_n(x)P_m(x)] = 0$ if $n \neq m$.

Thus, in the case of weak orthogonal polynomial sequence, we do not ensure that $\mathcal{L}[P_n^2(x)] \neq 0$ for all $n \geqslant 0$.
Note that if $\{P_n(x)\}_{n=0}^{\infty}$ is a weak orthogonal polynomial sequence with respect to $\mathcal{L}$, then $\{\alpha_n P_n(x)\}_{n=0}^{\infty}$ is also a weak orthogonal polynomial sequence with respect to the same $\mathcal{L}$ for any arbitrary complex sequence $\{\alpha_n\}_{n=0}^{\infty}$ such that $\alpha_n \neq 0$ for all $n \geqslant 0$. Therefore, without loss of generality, we will be considering *monic* polynomial sequences, i.e. such that each element of the sequence has a leading coefficient equal to 1.
Since we are going to work with weak orthogonal polynomial sequences, we need to slightly modify Favard's theorem, which plays an essential role in the theory of orthogonal polynomials (the original statement can be found in [27, Theorem 4.4]).

Theorem A.0.1 (Favard's theorem). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be arbitrary sequences of complex numbers and let $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the tree step recurrence given by:*

$$\begin{aligned} &P_{-1}(x) = 0, \qquad P_0(x) = 1, \\ &xP_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \qquad n \geqslant 0. \end{aligned} \tag{A.2}$$

Then there exists a unique moment functional $\mathcal{L}$ such that $\{P_n(x)\}_{n=0}^{\infty}$ is a weak orthogonal polynomial sequence with respect to $\mathcal{L}$.

Note that b_0 in the above three-step recurrence is unnecessary and can be chosen arbitrarily. However, to determine $\mathcal{L}$ uniquely we have to impose an additional condition on the value of $\mu_0 = \mathcal{L}[1]$.
Now, we will briefly sketch the proof of the modified Favard's theorem.

Proof of Theorem A.0.1. In view of (A.2), each $P_n(x)$ is a monic polynomial of degree n, $n \geqslant 0$. Fix $\mu_0 \in \mathbb{C}$. We define a moment functional $\mathcal{L}$ by the following conditions

$$\mathcal{L}[1] = \mu_0, \qquad \mathcal{L}[P_n(x)] = 0, \qquad n \in \mathbb{N}. \tag{A.3}$$

Hence $\mathcal{L}$ is uniquely determined up to the choice of μ_0. Moreover, by linearity, it is well-defined for any polynomial. Furthermore, $\mathcal{L}$ applied to the second equality in (A.2) gives

$$\mathcal{L}[xP_n(x)] = \mathcal{L}[P_{n+1}(x)] + a_n\mathcal{L}[P_n(x)] + b_n\mathcal{L}[P_{n-1}(x)] = 0, \qquad n \geqslant 2.$$

Successive application of $\mathcal{L}$ to the second line in (A.2) multiplied by x leads to

$$\mathcal{L}[x^2P_n(x)] = \mathcal{L}[xP_{n+1}(x)] + a_n\mathcal{L}[xP_n(x)] + b_n\mathcal{L}[xP_{n-1}(x)] = 0 \qquad \text{for } n \geqslant 3.$$

Consequently, repeating this procedure leads to

$$\mathcal{L}[x^kP_n(x)] = 0 \qquad \text{for } 0 \leqslant k < n,$$

and

$$\mathcal{L}[x^nP_n(x)] = b_n\mathcal{L}[x^{n-1}P_{n-1}(x)] \qquad \text{for } n \geqslant 1.$$

Since $\{P_n(x)\}_{n=0}^{\infty}$ are monic polynomials, by linearity of $\mathcal{L}$ we conclude that

$$\mathcal{L}[P_k(x)P_n(x)] = 0 \qquad \text{for } k \neq n,$$

and

$$\mathcal{L}[P_n^2(x)] = b_n\mathcal{L}[P_{n-1}^2(x)] \qquad \text{for } n \geqslant 0.$$

□

Iterating the last equality in the proof shows that for all $n \in \mathbb{N}$

$$\mathcal{L}[P_n^2(x)] = \mu_0 \cdot b_1 \cdot \ldots \cdot b_n, \tag{A.4}$$

where $\mu_0 = \mathcal{L}[1]$. Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence with respect to $\mathcal{L}$ under the additional assumptions that $\mu_0 \neq 0$ and $b_n \neq 0$ for all $n \geqslant 0$.
Moreover, the construction of $\mathcal{L}$ given in (A.3) implies that $\mathcal{L}$ is a zero operator if only $\mu_0 = 0$. This is not an interesting case, so we always assume that $\mu_0 \neq 0$ and further considerations are carried out under this additional assumption. Moreover, we say that $\mathcal{L}$ is *normalized* if $\mu_0 = 1$.
In probabilistic settings, nonnegative definite moment functionals play a significant role. Let us introduce this concept formally:

Definition A.0.4. We say that a moment functional $\mathcal{L}$ is *nonnegative definite* if for any nonnegative polynomial $f \geqslant 0$, i.e. $f(x) \geqslant 0$ for all $x \in \mathbb{R}$, we have

$$\mathcal{L}[f(x)] \geqslant 0.$$

If $\mathcal{L}$ is nonnegative definite, then it imposes some additional assumptions on coefficients $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ appearing in three-step recurrence (A.2). Indeed, (A.4) says then that $b_1 \cdot \ldots \cdot b_n \geqslant 0$ for all $n \geqslant 1$. Moreover,

$$0 \leqslant \mathcal{L}[x^{2k}] = \mu_{2k}, \qquad k \in \mathbb{N}_0.$$

The linearity of $\mathcal{L}$ implies

$$0 \leqslant \mathcal{L}\left[(x+1)^{2k}\right] = \sum_{l=0}^{2k} \binom{2k}{l} \mu_{2k-l}, \qquad k \in \mathbb{N}_0.$$

We can show inductively that μ_{2k+1} is real for $k \in \mathbb{N}_0$. Consequently, $\{\mu_k\}_{k=0}^{\infty}$ is a real sequence. As a result, according to (A.1), $\mathcal{L}[f(x)]$ is also real for any polynomial f with real coefficients. The three-step recurrence (A.2) multiplied by $P_n(x)$ yields

$$\mathcal{L}[xP_n^2(x)] = a_n \mathcal{L}[P_n^2(x)], \qquad n \in \mathbb{N}_0.$$

Let us start with P_0. Since $\mathcal{L}[P_0^2(x))] = \mathcal{L}[1] = \mu_0 > 0$, then

$$a_0 = \tfrac{\mathcal{L}[x]}{\mathcal{L}[1]} = \tfrac{\mu_1}{\mu_0} \in \mathbb{R}$$

and (A.2) implies that P_1 has real coefficients. Hence $\mathcal{L}[xP_1^2(x)] \in \mathbb{R}$. Moreover, if $\mathcal{L}[P_1^2(x)] = \mu_0 b_1 > 0$ (thus $b_1 > 0$), then we can show that a_1 is real.

Continuing in this manner, we see that the whole procedure works until $b_N = 0$ for some $N \geqslant 0$. Then we cannot infer anything about the remaining values of the sequence $\{a_n\}_{n=0}^{\infty}$. However, in this situation, all polynomials P_n, $n \geqslant N$, are divisible by P_N.

For the convenience of reference, we summarize the above considerations in the following remark:

Remark A.0.2. *If $\mathcal{L}$ is nonnegative definite and $\mu_0 > 0$, then for all $n \geqslant 0$*

$$b_1 \cdot \ldots \cdot b_n \geqslant 0 \tag{A.5}$$

and a_k is real for every $0 \leqslant k < \inf\{n \geqslant 1 : b_1 \cdot \ldots \cdot b_n = 0\}$. In particular, when $b_1 \ldots \cdot b_n > 0$ for all $n \geqslant 0$, then $\{a_n\}_{n=0}^{\infty}$ is a real sequence.

Under the conditions from Remark A.0.2 with $\mu_0 = 1$, [22, Theorem A.1.] says that $\mathcal{L}$ can be expressed as an integral with respect to some probability measure, i.e. there exists a probability measure ν such that for any polynomial f

$$\mathcal{L}[f(x)] = \int_{\mathbb{R}} f(x)\nu(\mathrm{d}x). \tag{A.6}$$

In particular, all moments of ν exist and are given by $\{\mu_k\}_{k=0}^{\infty}$. Moreover, if $N := \inf\{n \geqslant 1 : b_1 \cdot \ldots \cdot b_n = 0\} < \infty$, then ν is a discrete measure supported on real distinct zeros of the polynomial P_N. Hence, the form of the orthogonality measure ν is not affected by the polynomials P_n, $n \geqslant N$, and thus by the sequences $\{a_n\}_{n=N}^{\infty}$ and $\{b_n\}_{n=N}^{\infty}$.

Terminology

If a moment functional $\mathcal{L}$ is in the form (A.6), then we say that ν is an *orthogonality measure* for polynomials $\{P_n(x)\}_{n=0}^{\infty}$ or $\{P_n(x)\}_{n=0}^{\infty}$ are *orthogonal with respect to the measure ν*.

Throughout the thesis, we slightly abuse the definition of orthogonality by which we always mean the weak orthogonality given in Definition A.0.3. In particular, it will be of no interest to us whether $\mathcal{L}[P_n^2(x)] > 0$ for all $n \in \mathbb{N}$. We only require that $\mathcal{L}$ is normalized, i.e. $\mathcal{L}[1] = 1$.

Appendix B

List of Symbols

Here we list the symbols most frequently used in Chapters 1-6.

$\mathbb{N}$	set of natural numbers, i.e. $\mathbb{N} = \{1, 2, 3, , \dots\}$
$\mathbb{N}_0$	set of natural numbers including zero, i.e. $\mathbb{N}_0 = \{0, 1, 2, 3, , \dots\}$
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers

B.1. Quadratic harness

$QH(\eta, \theta; \sigma, \tau; q)$	quadratic harness with parameters η, θ, σ, τ and q, p. 17
$\mathbf{A}_t^+$	weak right infinitesimal generator, p. 23
$\mathbf{A}_t^-$	weak left infinitesimal generator, p. 23
$\mathbf{A}_t$	weak infinitesimal generator, p. 24
$\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$	family of orthogonal polynomials associated with infinitesimal generator, p. 27
$\{W_n(\cdot; x, t)\}_{n=0}^{\infty}$	auxiliary family of orthogonal polynomials, p. 37
$\mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q}$	normalized moment functional for $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$, p. 27
$\nu_{x,t,\eta,\theta,\sigma,\tau,q}$	probabilistic orthogonality measure for $\{\widetilde{W}_n(\cdot; x, t)\}_{n=0}^{\infty}$, p. 27

B.2. Algebra $\mathcal{Q}$

B.2.1. Objects of main interest

$\mathcal{Q}$	algebra Q of infinite sequences of polynomials, p. 31
$\mathcal{Q}_k$, $k = 0, 1, 2, \dots$	subspaces of $\mathcal{Q}$, p. 55
$\mathbb{A}_t$	element representing infinitesimal generator in $\mathcal{Q}$, p. 34
$\mathbb{H}_t$	pre-generator, p. 34
$\widetilde{\mathbb{H}}_t$	p. 38
$\widetilde{\mathbb{M}}_t$	p. 40
$\mathbb{S}(z,t)$, $\mathbb{S}$	element corresponding to the Jacobi matrix for $\{W_n(x;z,t)\}_{n=0}^{\infty}$, p. 38
$\mathbb{W}(z,t)$, $\mathbb{W}$	element corresponding to the polynomials $\{W_n(x;z,t)\}_{n=0}^{\infty}$, p. 38

B.2.2. Basic elements

$\mathbb{B}$	p. 73
$\mathbb{D}$, $\mathbb{F}$	shifting elements, p. 32
$\mathbb{D}_q$	q-derivative, p. 32
$\mathbb{E}$	identity, p. 31
$\mathbb{K}_1$, $\mathbb{K}_2$, $\mathbb{K}_3$	p. 70, p. 71
$\mathbb{Q}$	p. 67
$\mathbb{T}_1$, $\mathbb{T}_2$, $\mathbb{T}_3$	p. 68
$\mathbb{Z}$	p. 39
$\mathbb{Z}_0$, $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_3$	p. 65
$\mathbb{0}$	zero element, p. 32

B.2.3. Auxiliary elements

$\widetilde{\mathbb{P}}$, $\widehat{\mathbb{P}}$, $\mathbb{P}$	p. 74
$\widetilde{\mathbb{P}}_1$, $\widetilde{\mathbb{P}}_2$, $\widetilde{\mathbb{P}}_3$	components of $\widetilde{\mathbb{P}}$, p. 74
$\widehat{\mathbb{P}}_1$, $\widehat{\mathbb{P}}_2$, $\widehat{\mathbb{P}}_3$	components of $\widehat{\mathbb{P}}$, p. 74
$\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$	components of $\mathbb{P}$, p. 74
$\mathbb{U}$	p. 74
$\widetilde{\mathbb{X}}$, $\mathbb{X}$	p. 75
$\widetilde{\mathbb{X}}_2$, $\widetilde{\mathbb{X}}_3$	components of $\widetilde{\mathbb{X}}$, p. 75
$\mathbb{X}_2$, $\mathbb{X}_3$	components of $\mathbb{X}$, p. 75
$\mathbb{Y}$	p. 74

B.2.4. Operators acting on elements of $\mathcal{Q}$

$\mathcal{R}$	p. 38
$\mathcal{S}$	p. 58
$\mathcal{T}$	p. 60
$[\cdot, \cdot]$	commutator, p. 61

Bibliography

[1] Agoitia-Hurtado, M. (2017). *Time-inhomogeneous polynomial processes in electricity spot price models*. PhD thesis, University of Freiburg.

[2] Agoitia Hurtado, M. F. D. C. and Schmidt, T. (2020). Time-inhomogeneous polynomial processes. *Stoch. Anal. Appl.*, 38(3):527–564.

[3] Anshelevich, M. (2003). Free martingale polynomials. *J. Funct. Anal.*, 201(1):228–261.

[4] Anshelevich, M. (2004). Appell polynomials and their relatives. *Int. Math. Res. Not.*, (65):3469–3531.

[5] Anshelevich, M. (2013). Generators of some non-commutative stochastic processes. *Probab. Theory Related Fields*, 157(3-4):777–815.

[6] Benth, F., Detering, N., and Kruhner, P. (2020). Abstract polynomial processes. arXiv: 2010.02483.

[7] Benth, F. E., Detering, N., and Krühner, P. (2021). Independent increment processes: a multilinearity preserving property. *Stochastics*, 93(6):803–832.

[8] Berestycki, N., Powell, E., and Ray, G. (2021). $(1+\varepsilon)$ moments suffice to characterise the GFF. *Electron. J. Probab.*, 26:Paper No. 44, 25.

[9] Biane, P. (1996). Quelques propriétés du mouvement brownien non-commutatif. *Astérisque*, (236):73–101.

[10] Biane, P. (1998). Processes with free increments. *Math. Z.*, 227(1):143–174.

[11] Biane, P. and Speicher, R. (1998). Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probab. Theory Related Fields*, 112(3):373–409.

[12] Billingsley, P. (2012). *Probability and measure*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., Hoboken, NJ, anniversary edition. With a foreword by Steve Lalley and a brief biography of Billingsley by Steve Koppes.

[13] Böttcher, B. (2014). Feller evolution systems: generators and approximation. *Stoch. Dyn.*, 14(3):1350025, 15.

[14] Bożejko, M., Kümmerer, B., and Speicher, R. (1997). q-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.*, 185(1):129–154.

[15] Bryc, W. (2010). Markov processes with free-Meixner laws. *Stochastic Process. Appl.*, 120(8):1393–1403.

[16] Bryc, W. (2012). Quadratic harnesses from generalized beta integrals. 96:67–79.

[17] Bryc, W. and Matysiak, W. (2016). Wilson's $6-j$ laws and stitched Markov processes. *Theory Probab. Appl.*, 60(2):225–245.

[18] Bryc, W., Matysiak, W., and Wesołowski, J. (2007). Quadratic harnesses, q-commutations, and orthogonal martingale polynomials. *Trans. Amer. Math. Soc.*, 359(11):5449–5483.

[19] Bryc, W., Matysiak, W., and Wesołowski, J. (2008). The bi-Poisson process: a quadratic harness. *Ann. Probab.*, 36(2):623–646.

[20] Bryc, W., Matysiak, W., and Wesołowski, J. (2011). Free quadratic harness. *Stochastic Process. Appl.*, 121(3):657–671.

[21] Bryc, W. and Wesołowski, J. (2005). Conditional moments of q-Meixner processes. *Probab. Theory Related Fields*, 131(3):415–441.

[22] Bryc, W. and Wesołowski, J. (2010). Askey-Wilson polynomials, quadratic harnesses and martingales. *Ann. Probab.*, 38(3):1221–1262.

[23] Bryc, W. and Wesołowski, J. (2014). Infinitesimal generators of q-Meixner processes. *Stochastic Process. Appl.*, 124(1):915–926.

[24] Bryc, W. and Wesołowski, J. (2015). Infinitesimal generators for a class of polynomial processes. *Studia Math.*, 229(1):73–93.

[25] Bryc, W. and Wesołowski, J. (2017). Asymmetric simple exclusion process with open boundaries and quadratic harnesses. *J. Stat. Phys.*, 167(2):383–415.

[26] Bryc, W., Wesołowski, J., and Zięba, A. (2022). Flows in near algebras with applications to harnesses. *Colloq. Math.*, 170(2):211–238.

[27] Chihara, T. S. (1978). *An introduction to orthogonal polynomials.* Gordon and Breach Science Publishers, New York-London-Paris,,.

[28] Cohen, J. M. and Trenholme, A. R. (1984). Orthogonal polynomials with a constant recursion formula and an application to harmonic analysis. *J. Funct. Anal.*, 59(2):175–184.

[29] Cuchiero, C. (2011). *Affine and polynomial processes.* PhD thesis, ETH Zürich.

[30] Cuchiero, C., Keller-Ressel, M., and Teichmann, J. (2012). Polynomial processes and their applications to mathematical finance. *Finance Stoch.*, 16(4):711–740.

[31] Cuchiero, C., Larsson, M., and Svaluto-Ferro, S. (2019). Probability measure-valued polynomial diffusions. *Electron. J. Probab.*, 24:Paper No. 30, 32.

[32] Cuchiero, C. and Svaluto-Ferro, S. (2021). Infinite-dimensional polynomial processes. *Finance Stoch.*, 25(2):383–426.

[33] Dozzi, M. (1981). Two-parameter harnesses and the Wiener process. *Z. Wahrsch. Verw. Gebiete*, (no. 4):507–514.

[34] Hammersley, J. M. (1967). Harnesses. In *Proc. Fifth Berkeley Sympos. Mathematical Statistics and Probability (Berkeley, Calif., 1965/66), Vol. III: Physical Sciences*, pages 89–117.

[35] Hoyle, E. and Menguturk, L. A. (2020). Generalised Liouville processes and their

properties. *J. Appl. Probab.*, 57(4):1088–1110.

[36] Ismail, M. E. H. (2009). *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge. With two chapters by Walter Van Assche, With a foreword by Richard A. Askey, Reprint of the 2005 original.

[37] Jamiołkowska, M. (2009). Bi-Pascal process: definition and properties. Master's thesis, Warsaw University of Technology.

[38] Mansuy, R. and Yor, M. (2005). Harnesses, Lévy bridges and *Monsieur Jourdain*. *Stochastic Process. Appl.*, 115(2):329–338.

[39] Schoutens, W. (2000). *Stochastic processes and orthogonal polynomials*, volume 146 of *Lecture Notes in Statistics*. Springer-Verlag, New York.

[40] Szabłowski, P. J. (2014). A few remarks on quadratic harnesses. *J. Difference Equ. Appl.*, 20(4):586–609.

[41] Szabłowski, P. J. (2015). On Markov processes with polynomial conditional moments. *Trans. Amer. Math. Soc.*, 367(12):8487–8519.

[42] Szabłowski, P. J. (2018). Markov processes, polynomial martingales and orthogonal polynomials. *Stochastics*, 90(1):61–77.

[43] Wesołowski, J. (1993). Stochastic processes with linear conditional expectation and quadratic conditional variance. *Probab. Math. Statist.*, 14(1):33–44.

[44] Wesołowski, J. and Zięba, A. (2023). Infinitesimal generators for a family of polynomial processes – an algebraic approach. arXiv: 2305.00198.