Synopsis

At the beginning elements of the theory of reproducing kernel Hilbert spaces will be recalled. It will be shown that existence of the reproducing kernel is equivalent to continuity of the functionals of point evaluation. The proof that if $\mathcal{H}$ is a reproducing kernel Hilbert space of functions defined on U, then in any set

$$\{f \in \mathcal{H} \mid f(z) = 1\}, z \in U$$

if non-empty, there is exactly one element with minimal norm, will be given.

The second chapter will be devoted to the Hilbert spaces of square-integrable functions which are the kernel (in the algebraic sense) of some elliptic operator. It will be shown that such a space is a reproducing kernel Hilbert space. Then we will prove that the reproducing kernel of that space depends in a continuous way on a weight of integration, i.e. on a deformation of an inner product. Convergence of weights only almost everywhere will be needed. Next we will generalize Ramadanov theorem, i.e. we will show that the reproducing kernel of such a space depends in continuous way on a domain of integration, i.e. on a domain on which our functions are defined. It will be done in three different ways for the case an increasing sequence of domains. Moreover sufficient condition for the case of decreasing sequence of domains will be given.

Particular case of such a Hilbert space is Hilbert space of square-integrable and harmonic functions. In such a case it will be shown that if only an inverse of a weight of integration is integrable in some positive power, then the reproducing kernel of the corresponding weighted Hilbert space exists. Moreover an example of a weight for which there is no reproducing kernel of such a space will be given.

By the minimal norm property of the reproducing kernel recalled in the first chapter, we will conclude that in the set of square-integrable solutions of an elliptic equation, which take value at some given point equal to c, if non-empty, there is exactly one element with minimal norm. Moreover such an element depends in continuous way on a weight and domain of integration in a precisely defined sense.

The third chapter will be devoted to the weighted kernels of Szegö type. We will give sufficient conditions for a weight of integration in order for the reproducing kernel of the weighted Szegö space to exist. In particular it will be shown that if an inverse of a weight of integration is integrable, then there exists the reproducing kernel of the corresponding weighted Szegö space. The case of domains with non-connected boundaries will be also considered. Moreover we will give an example of a weight for the unit ball for which there is no reproducing kernel of the corresponding space. Using biholomorphisms we will prove that such weights exist for a large class of domains.

Then we will prove that Szegö kernel depends in a continuous way on a weight of integration. Pasternak's theorem on dependence of the orthogonal projector on a deformation of an inner product will be used in the proof. Finally it will be shown how weighted Szegö kernel can be used to prove general theorems of complex analysis.

Keywords: Reproducing kernel Hilbert space, functional of point evaluation, reproducing kernel, elliptic operator, elliptic equation, minimal solution, Szegö kernel, admissible weights, weights of integration, dependence on parameters, continuous dependence, Ramadanov theorem, continuous dependence on a weight of integration.