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REPORT ON THE PH.D. THESIS "INFINITESIMAL GENERATORS OF QUADRATIC HARNESSES" BY AGNIESZKA ZIĘBA

Ms. Zięba has produced a substantial and important thesis. She completely solves a natural problem, combining a variety of techniques drawn from algebra, functional analysis, operator theory, and the study of Markov processes. I was especially impressed by her attention to detail. At multiple places throughout the thesis, she proves a theorem under some assumptions, and then takes the time to provide explicit counterexamples for when the assumptions are not satisfied. In fact, the thesis is so complete that it is hard to think of follow-up work. Nevertheless, since quadratic harnesses are known to be related to q-special functions, asymmetric exclusion process, and umbral calculus, this research is likely to lead to further exploration of these connections.

Since, as mentioned above, the thesis is very detailed, it is hard to summarize its contents. Here is a very brief description of its main result.

For a random process $(X_t)_{t\geq 0}$, denote by $\mathcal{F}_{s,u}$ the σ-algebra generated by X_t for $t \leq s$ and $t \geq u$. The process is a quadratic harness if its conditional expectation and conditional variance with respect to $\mathcal{F}_{s,u}$ are linear, respectively, quadratic functions of X_s and X_u. The key theorem of Bryc, Matysiak, Wesołowski (2007), updated in Bryc, Wesołowski, Zięba (2022), is that, under natural normalization and independence assumptions, a quadratic harness is characterized by five parameters

$$\eta, \theta \in \mathbb{R}, \quad \sigma, \tau \geq 0, \quad q \leq 1 + 2\sqrt{\sigma\tau}.$$

Despite being determined by only five parameters, this class contains many of the processes appearing in various applications. One can compare it to the (also five-parameter) family of Askey-Wilson polynomials, which form a small but crucial subset of all orthogonal polynomial families.

Under the additional assumptions

$$-1 \leq q \leq 1 - 2\sqrt{\sigma\tau}, \quad 0 \leq \sigma\tau < 1,$$

the corresponding quadratic harness is know to exist. An additional assumption made in the thesis, which may not follow from those above, is that all moments of the process are finite. The main result of the thesis is that the generator of the quadratic harness X_t, as a Markov process, has the form

$$A_t f(x) = \frac{1 + \eta x + \sigma x^2}{1 + \sigma t} \mathcal{L}_{x,t,\eta,\theta,\sigma,\tau,q} \left(\frac{\partial}{\partial x} \frac{f(x) - f(y)}{x - y} \right),$$

where the linear functional $\mathcal{L}$ is described explicitly through its orthogonal polynomials; under additional assumptions the linear functional arises from a measure.

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A variety of partial results of this type were known previously, all of which are unified by Ms. Zięba's work. Notably, all earlier constructions required some of the parameters to vanish or satisfy simple relations. She also extends slightly the range of the parameters for which the quadratic harness is known to exist.

Again, the description above omits many of the important technical details.

While the general outline of the argument follows the path laid our in the earlier work by Bryc and Wesołowski, Ms. Zięba was able to overcome major technical difficulties. She demonstrated expertise in algebraic techniques, deep knowledge of functional analysis, and familiarity with a large volume of literature.

I would make one bibliographic comment. The key operation on general polynomial sequences, defined on page 31, was introduced by Bryc and Wesolowski. However its provenance goes back much further. Appell defined this operation in 1880, but only on polynomial sequences (P_n) such that $P_n' = nP_{n-1}$ (such sequences actually form a commutative subalgebra). The operation is typically called "umbral composition".

In conclusion, Ms. Zięba has done excellent work, and her thesis easily satisfies and surpasses the requirements.